Lecture I: Dirac Notation

• To describe a physical system, QM assigns a complex number (‘amplitude’) to each distinct available physical state.
  - (Or alternately: two real numbers)
  - What is a ‘distinct physical state’?

• Consider a system with M distinct available states
  - The 2M real numbers can be viewed as a vector in an 2M-dimensional real-valued vector space
  - Or alternatively as a vector in an M-dimensional complex-valued vector space
  - We will refer to this abstract vector space as ‘Hilbert Space’ or ‘state space’
  - Any vector in this space corresponds to a possible quantum-mechanical state. The number of such quantum states is uncountable infinity

• Just as calculus provides the mathematical basis for Classical Mechanics, the mathematical basis for QM is linear algebra
  - Vectors, matrices, eigenvalues, rotations, etc... are key concepts

Various common vector notations:

1. Vector notation: \( \vec{r}(t) \)
   - Just a name, an abstraction that refers to something physical
2. Unit vectors: \( \vec{r}(t) = r_1(t)\vec{e}_1 + r_2(t)\vec{e}_2 + r_3(t)\vec{e}_3 \)
   - Unit vectors are predefined in physical terms
   - Components are projections onto unit vectors
   - Unit vectors are orthonormal
3. Column vector:
   - Unit vectors are implied
   \[ \vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix} \]

‘Dirac notation’:

- Just new symbols for same concepts ‘ket’
  \( \vec{r}(t) \rightarrow |\psi(t)\rangle, \langle \varphi(t) |, |\Psi(t)\rangle, ... \) ‘bra’
  \( \vec{r}^T(t) \rightarrow \langle \psi(t) |, |\varphi(t)\rangle, \langle \Psi(t) |, ... \)
  \( \vec{e}_j \rightarrow |j\rangle, |n\rangle, |a_n\rangle, |r\rangle, |p\rangle, |n,m\rangle, |E_n\rangle, |E_n,m\rangle, ... \)
\( \vec{r}(t) = r_1(t)\vec{e}_1 + r_2(t)\vec{e}_2 + r_3(t)\vec{e}_3 \rightarrow |\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle \)
  \( \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \langle a | b \rangle \) ‘inner product’
  \( r_j(t) = \vec{e}_j \cdot \vec{r}(t) \rightarrow c_j(t) = \langle j | \psi(t) \rangle \)
Added catch since QM vectors are complex

- Transpose operation replaced by 'Hermitian conjugation' or 'dagger' operation
  \[ \langle b | a \rangle = \langle a | b \rangle^* \]
  
  - '†' is transpose plus complex conjugation
  \[ \tilde{\bar{r}}^T = (\tilde{\bar{r}})^T \rightarrow \langle \bar{\psi} | = (|\bar{\psi}\rangle)^\dagger \]

\[ \tilde{r} = r_1 \tilde{e}_1 + r_2 \tilde{e}_2 + \ldots + \tilde{r}_M \tilde{e}_M \]
\[ = \tilde{e}_1 (\tilde{e}_1 \cdot \tilde{r}) + \tilde{e}_2 (\tilde{e}_2 \cdot \tilde{r}) + \ldots + \tilde{e}_M (\tilde{e}_M \cdot \tilde{r}) \]

- Projectors and Closure relations:
  \[ |\psi\rangle = c_1|1\rangle + c_2|1\rangle + \ldots + c_M|M\rangle \]
  \[ = |1\langle 1|\psi\rangle + |2\rangle\langle 2|\psi\rangle + \ldots + |M\rangle\langle M|\psi\rangle \]
  \[ = (|1\rangle\langle 1| + |2\rangle\langle 2| + \ldots + |M\rangle\langle M|)\psi \]

- This proves the `closure relation': \[ \sum_{j=1}^{M} |j\rangle\langle j| = 1 \]
  
  The summation is over a complete set of unit vectors that spans any Hilbert sub-space is equal to the identity operator in that sub-space
  
  - The entire Hilbert space is a trivial sub-space

- Norm of a vector:
  - a.k.a. magnitude, length
  \[ \|r\| = \sqrt{\bar{r} \cdot \bar{r}} \]
  \[ = \sqrt{\bar{r}^T \bar{r}} \rightarrow \|\psi\| = \sqrt{\langle \psi | \bar{\psi} \rangle} \]

- To compute the norm in terms of the components along a set of orthogonal unit vectors:
  
  - Insert the identity
  \[ \langle \psi | = \langle \psi | \right \langle j \rangle |j\rangle \]
  \[ = \left \langle \psi \left | \sum_{j=1}^{M} |j\rangle\langle j| \right \langle j \right | \psi \right \rangle \]
  \[ = \sum_{j=1}^{M} \langle \psi | j \rangle \langle j | \psi \rangle \]
  \[ = \sum_{j=1}^{M} c_j^* c_j = \sum_{j=1}^{M} |c_j|^2 \]

  Old notation:
  \[ \bar{r} \cdot \tilde{r} = r_1^2 + r_2^2 + \ldots \]
  \[ = \sum_{j} r_j^2 \]
Avoid being confused by implied meanings of various symbols

- To avoid confusion, keep in mind that \(| \psi \rangle \) indicates a Hilbert-space vector, the \( \psi \) in \(| \psi \rangle \) is just a label
  - We could call it anything
    - \(| \psi \rangle, |\phi \rangle, |3 \rangle, |\text{Alice} \rangle \)
  - We just need to clearly define our labels

  - "let \(| \psi(t) \rangle \) be the state of our system at time \( t \)."

  - "let \(| x \rangle \) be the state in which the particle lies at position \( x \)"
    - Here \( x \) is a placeholder which could take on any numerical value. I.e. defining the state \(| x \rangle \) as above actually defines an infinite set of vectors, one for each point on the real axis.
    - This is exactly how the symbol \( x \) is used when you say \( f(x) = \cos(x) \)

- \( |j \rangle \) be the state in which our system is in the \( j \)th quantized energy level.
  - Here \( j \) is a placeholder for an arbitrary integer

Summary

- There are 'ket's and 'bra's:
  - kets: \(| \psi \rangle \)
    - A ket is a vector in an \( M \) dimensional Hilbert space, where \( M \) is the number of distinct physical states of a system
  - bras: \( \langle \psi | \)
    - A bra is a transposed, conjugated ket

- Put a bra and a ket together to get a c-number
  - \( \langle \psi | \psi \rangle := \text{a } c\text{-number} \)
    - \( c\text{-number} := \text{complex number} \)

- Unit vectors:
  - An \( M \) dimensional Hilbert space is spanned by \( M \) orthonormal unit vectors
    - \( \{ |j \rangle \}_{j=1}^{M} \) (\( \{ \} = \text{the set of} \)
    - \( \langle j | k \rangle = \delta_{jk} \) (\( \delta_{jk} \) is 'Kronecker delta function')
      - \( 1 \text{ if } j=k \)
      - \( 0 \text{ else} \)
    - Closure relation:
      \[ \sum_{j=1}^{M} |j \rangle \langle j | = 1 \]