QUANTUM MECHANICS SUBJECT EXAM
Spring 2008

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1. POSTULATES OF QUANTUM MECHANICS: A certain quantum system is governed by the Hamiltonian $H = \sum_{n=0}^{\infty} E_n |n\rangle\langle n|$, where the state $|n\rangle$ is the n^{th} energy eigenstate of the system, with eigenvalue E_n. Assume that the system is prepared in the initial state $|\psi(t_0)\rangle = \frac{1}{\sqrt{6}}\left[|1\rangle + 2|2\rangle + |3\rangle\right]$, where the states $|1\rangle$, $|2\rangle$ and $|3\rangle$ refer to the 1st 2nd and 3rd energy eigenstates. At time t_1 a measurement is made of the total energy of the system.

   (a) Taking the initial state into account, what are the possible results from the measurement.

   (b) What is the probability of obtaining each possible result?

   (c) For each possible result, what is the normalized state of the system after the measurement?

   Now consider a system of two spin-1/2 particles prepared in the state

   $$|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle,$$

   where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. While the particles are in this state, the operator S^2 is measured, where $\vec{S} = \vec{S}_1 + \vec{S}_2$ is the total spin.

   (d) What are the possible results of the measurement?

   (e) What are the probabilities for getting each possible result? (In terms of a, b, c, and d)

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2. SCATTERING IN 1-D: A particle with energy E is incident from the left onto a combination of a step potential and a delta-potential. Thus the total potential the particle sees is $V(x) = V_1(x) + V_2(x)$, where $V_1(x) = U_0$ for $x > 0$ and is zero otherwise, and $V_2(x) = V_0\delta(x)$. Compute the probability that the particle is **reflected**. Give the answer in terms of $k = \sqrt{2mE}/\hbar$ and $K = \sqrt{2m(E - U_0)}/\hbar$, as well as the constants m, V_0 and $\hbar$. You can assume that $E > U_0$.

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3. PERTURBATION THEORY:

Let

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

and

$$V = V_0 \delta(x).$$

Denote the bare eigenstates by $|n\rangle$, with $H_0|n\rangle = \hbar\omega_n|n\rangle$; $n = 0, 1, 2, \ldots$.

(a) For this system, what is the energy of the nth eigenstate of H_0?

(b) Based on symmetry considerations, which bare levels will not be shifted at all by this perturbation?

(c) Use perturbation theory to find the ground state energy of $H = H_0 + \lambda V$ to second order in λ, and the ground-state wavefunction to first order in λ.

Tip: You can write"$\phi_n(0)$" for the bare trap eigenfunctions at $x = 0$, rather than finding an exact expression.

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4. THE THREE DIMENSIONAL RIGID ROTOR: Consider a particle in three dimensions which is constrained to move on the surface of a sphere of radius R. The classical kinetic energy of a particle moving on the sphere with angular velocity ω about some axis is $I\omega^2/2$, where $I = mR^2$ is the moment of inertia. The kinetic energy may be written $L^2/2I$, where $\vec{L}$ is the (classical) angular momentum. The corresponding quantum Hamiltonian is $H = L^2/2I$, where $\vec{L}$ is now the angular momentum operator.

(a) What are the energy levels of the quantum mechanical rigid rotor? What is the degeneracy of each energy level?

(b) A convenient basis for the position of the particle is given by $|\theta\phi\rangle$, where θ and ϕ are the angles in spherical polar coordinates. What are the energy eigenfunctions in this basis?

Now, suppose the particle is charged and the sphere is in a uniform magnetic field $\vec{B} = B_z\hat{k}$. The Hamiltonian now becomes

$$H = \frac{\text{\L}^2}{2I} + \gamma B_z L_z.$$

(c) What are the energy levels and eigenfunctions now? What is the degeneracy of each level?

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5. THE BORN SERIES:
The Lippman-Schwinger equation is

$$|\psi\rangle = |\psi_0\rangle + GV|\psi\rangle,$$

(a) Starting from the Lippman-Schwinger equation, derive the Born series expansion for the full solution, $|\psi\rangle$.

(b) The T-matrix is defined via $|\psi\rangle = |\psi_0\rangle + GT|\psi_0\rangle$. Combine this with your result from (a) and derive the Born series expansion for T.

(c) The scattering amplitude, $f(\vec{k}', \vec{k})$ is defined via

$$f(\vec{k}', \vec{k}) = -\frac{(2\pi)^2 M}{\hbar^2} \langle \vec{k}' | T | \vec{k} \rangle.$$

Using the first-order Born approximation for T, find the scattering amplitude for the Gaussian potential $V(\vec{r}) = V_0 e^{-(r/R)^2}$.
HINT: $\int_{=\infty}^{\infty} dx\, e^{iqx} e^{-ax^2} = \sqrt{\frac{\pi}{a}} e^{-q^2/4a^2}$.

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6. HEISENBERG EQUATIONS OF MOTION: The quantum state in the Heisenberg picture is related to the state in the Schrödinger picture by

$$|\psi_H\rangle = e^{iHt/\hbar}|\psi_S(t)\rangle.$$

(a) Let $\hat{O}$ be an arbitrary operator. How must $\hat{O}_H(t)$ be related to $\hat{O}_S$ so that the expectation value $\langle\hat{O}\rangle$ is the same in both pictures?

(b) Use the previous result to derive the Heisenberg equation of motion for $\hat{O}_H(t)$.

A certain nonlinear crystal absorbs photons of energy $2\hbar\omega$ and emits two photons of energy $\hbar\omega$. When this crystal is placed between two high-quality mirrors, and pumped by a laser with frequency 2ω, the buildup of light at frequency ω is governed by the Hamiltonian

$$\hat{H} = \hbar\chi(\hat{A}\hat{A} + \hat{A}^\dagger\hat{A}^\dagger),$$

where $\hat{A}$ and $\hat{A}^\dagger$ are the operators which describe the destruction and creation of a photon in the resonator mode. They satisfy the usual bosonic commutation relation

$$[\hat{A}, \hat{A}^\dagger] = 1.$$

(c) Using the Heisenberg picture, find the equations of motion for the operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$.

(d) Solve the equations to relate $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ to their initial values $\hat{A}(0)$ and $\hat{A}^\dagger(0)$.