Chapter 8

Rotational Equilibrium and Rotational Dynamics

Torque

- Torque, \( \tau \), is tendency of a force to rotate object about some axis
  \[ \tau = Fd \]
- \( F \) is the force
- \( d \) is the lever arm (or moment arm)
- Units are \( \text{Newton-m} \)

Torque is vector quantity

- Direction determined by axis of twist
- Perpendicular to both \( r \) and \( F \)
- Clockwise torques point into paper. Defined as negative
- Counter-clockwise torques point out of paper. Defined as positive

Non-perpendicular forces

\[ \tau = F r \sin \phi \]

\( \phi \) is the angle between \( F \) and \( r \)

Torque and Equilibrium

\[ \sum F_x = 0 \quad \text{and} \quad \sum F_y = 0 \]
- Forces sum to zero (no linear motion)

\[ \sum \tau = 0 \]
- Torques sum to zero (no rotation)
**Center of Gravity**

- Gravitational force acts on all points of an extended object.
- However, it can be considered as one net force acting on one point, the center-of-gravity, X.

\[
\sum_i (m_i g)x_i = \sum_i \frac{m_i}{\sum m_i} g \sum_i m_i x_i = M_g X, \text{ where } X = \frac{\sum m_i x_i}{\sum m_i}
\]

**Example 8.1**

Given \( M = 120 \text{ kg} \).
Neglect the mass of the beam.

a) Find the tension in the cable

b) What is the force between the beam and the wall

a) \( T = 824 \text{ N} \)  
   b) \( f = 353 \text{ N} \)

**Another Example**

Given: \( W = 50 \text{ N}, L = 0.35 \text{ m}, x = 0.03 \text{ m} \)
Find the tension in the muscle

\[ F = 583 \text{ N} \]

**Example 8.2**

Given: \( x = 1.5 \text{ m}, L = 5.0 \text{ m} \),
\( \text{w}_{\text{beam}} = 300 \text{ N}, \text{w}_{\text{man}} = 600 \text{ N} \)
Find: \( T \)

\[ T = 413 \text{ N} \]
Example 8.3
Consider the 400-kg beam shown below.
Find \( T_R \)

\[ T_R = 1121 \text{ N} \]

Example 8.4a
Given:
- \( W_{\text{beam}} = 300 \)
- \( W_{\text{box}} = 200 \)
Find:
- \( T_{\text{left}} \)
What point should I use for torque origin?
A B C D

Example 8.4b
Given:
- \( T_{\text{left}} = 300 \)
- \( T_{\text{right}} = 500 \)
Find:
- \( W_{\text{beam}} \)
What point should I use for torque origin?
A B C D

Example 8.4c
Given:
- \( W_{\text{beam}} = 300 \)
- \( W_{\text{box}} = 200 \)
Find:
- \( T_{\text{right}} \)
What point should I use for torque origin?
A B C D

Example 8.4d
Given:
- \( T_{\text{left}} = 250 \)
- \( T_{\text{right}} = 400 \)
Find:
- \( W_{\text{box}} \)
What point should I use for torque origin?
A B C D

Example 8.4e
Given:
- \( T_{\text{left}} = 250 \)
- \( W_{\text{beam}} = 250 \)
Find:
- \( W_{\text{box}} \)
What point should I use for torque origin?
A B C D
Example 8.5 (skip)
A 80-kg beam of length $L = 100$ cm has a 40-kg mass hanging from one end. At what position $x$ can one balance them beam at a point?

\[ x = 66.67 \text{ cm} \]

**Torque and Angular Acceleration**

Analogous to relation between $F$ and $a$

\[ F = ma, \quad \tau = I\alpha \]

**Moment of Inertia**

- Mass analog is moment of inertia, $I$
  \[ I = \sum m_i r_i^2 \]
- $r$ defined relative to rotation axis
- SI units are $\text{kg m}^2$

**More About Moment of Inertia**

- $I$ depends on both the mass and its distribution.
- If mass is distributed further from axis of rotation, moment of inertia will be larger.

**Moment of Inertia of a Uniform Ring**

- Divide ring into segments
- The radius of each segment is $R$
  \[ I = \sum m_i r_i^2 = MR^2 \]
Example 8.6
What is the moment of inertia of the following point masses arranged in a square?

a) about the x-axis?

b) about the y-axis?

c) about the z-axis?

a) 0.72 kg m²  
b) 1.08 kg m²  
c) 1.8 kg m²

Other Moments of Inertia

- cylindrical shell: \( I = MR^2 \)  
- solid cylinder: \( I = \frac{1}{2} MR^2 \)  
- rod about center: \( I = \frac{1}{12} ML^2 \)  
- rod about end: \( I = \frac{1}{3} ML^2 \)  
- spherical shell: \( I = \frac{2}{3} MR^2 \)  
- solid sphere: \( I = \frac{2}{5} MR^2 \)  

Example 8.7
Treat the spindle as a solid cylinder.

a) What is the moment of Inertia of the spindle? (M=5.0 kg, R=0.6 m)

b) If the tension in the rope is 10 N, what is the angular acceleration of the wheel?

c) What is the acceleration of the bucket?

d) What is the mass of the bucket?

a) 0.9 kg m²  
b) 6.67 rad/s²  
c) 4 m/s²  
d) 1.72 kg

Example 8.8 (skip)

A cylindrical space station of (R=12, M=3400 kg) has moment of inertia 0.75 MR². Retro-rockets are fired tangentially at the surface of space station and provide impulse of 2.9x10⁴ N·s.

a) What is the angular velocity of the space station after the rockets have finished firing?

b) What is the centripetal acceleration at the edge of the space station?

a) \( \omega = 0.948 \) rad/s  
b) a=10.8 m/s²

Example 8.9

A 600-kg solid cylinder of radius 0.6 m which can rotate freely about its axis is accelerated by hanging a 240 kg mass from the end by a string which is wrapped about the cylinder.

a) Find the linear acceleration of the mass.

b) What is the speed of the mass after it has dropped 2.5 m?

a) \( 4.36 \) m/s²  
b) 4.67 m/s
Rotational Kinetic Energy

Each point of a rigid body rotates with angular velocity $\omega$.

$$ KE = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i r_i^2 \omega^2 $$

$$ KE = \frac{1}{2} I \omega^2 $$

Including the linear motion

$$ KE = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 $$

KE of center-of-mass motion

Example 8.10

What is the kinetic energy of the Earth due to the daily rotation?

Given: $M_{\text{Earth}} = 5.98 \times 10^{24}$ kg, $R_{\text{Earth}} = 6.36 \times 10^6$ m.

KE due to rotation

$$ 2.56 \times 10^{29} \text{ J} $$

Example 8.11

A solid sphere rolls down a hill of height 40 m. What is the velocity of the ball when it reaches the bottom? (Note: We don’t know R or M!)

$$ v = 23.7 \text{ m/s} $$

Example 8.12a

The winner is:

A) Hollow Cylinder
B) Solid Cylinder

Example 8.12b

The winner is:

A) Hollow Cylinder
B) Sphere
Example 8.12c
The winner is:
A) Sphere
B) Solid Cylinder

Example 8.12d
The winner is:
A) Solid Cylinder
B) Monster

Example 8.12e
The winner is:
A) Sphere
B) Mountain Dew

Angular Momentum

\[ L = I \omega \]
\[ L = mvr = m\omega r^2 \]

Analogy between \( L \) and \( p \)

<table>
<thead>
<tr>
<th>Angular Momentum</th>
<th>Linear momentum</th>
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</thead>
<tbody>
<tr>
<td>( L = I \omega )</td>
<td>( p = mv )</td>
</tr>
<tr>
<td>( \tau = \Delta L/\Delta t )</td>
<td>( F = \Delta p/\Delta t )</td>
</tr>
<tr>
<td>Conserved if no net outside torques</td>
<td>Conserved if no net outside forces</td>
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</table>

Angular Momentum and Kepler's 2nd Law

- For central forces, e.g. gravity, \( \tau = 0 \) and \( L \) is conserved.
- Change in area in \( \Delta t \) is:

\[ \Delta A = \frac{1}{2} r(v_{\perp} \Delta t) \]
\[ L = mrv_{\perp} \]
\[ \frac{\Delta A}{\Delta t} = \frac{1}{2m} L \]
Example 8.13

A 65-kg student sprints at 8.0 m/s and leaps onto a 110-kg merry-go-round of radius 1.6 m. Treating the merry-go-round as a uniform cylinder, find the resulting angular velocity. Assume the student lands on the merry-go-round while moving tangentially.

\[ \omega = 2.71 \text{ rad/s} \]

Example 8.14

Two twin ice skaters separated by 10 meters skate without friction in a circle by holding onto opposite ends of a rope. They move around a circle once every five seconds. By reeling in the rope, they approach each other until they are separated by 2 meters.

a) What is the period of the new motion?

\[ T_f = T_o / 25 = 0.2 \text{ s} \]

b) If each skater had a mass of 75 kg, what is the work done by the skaters in pulling closer?

\[ W = 7.11 \times 10^5 \text{ J} \]