your name____

Physics 321 Exam #1 - Wednesday, Oct. 10, 2018

FYI:
Some integrals:

$$\begin{aligned}
\int_0^x \frac{dy}{\sqrt{1-y^2}} &= \sin^{-1}(x), \\
\int_0^x \frac{dy}{\sqrt{1+y^2}} &= \sinh^{-1}(x), \\
\int_0^x \frac{dy}{1+y^2} &= \tan^{-1}(x), \\
\int_0^x \frac{dy}{1-y^2} &= \tanh^{-1}(x), \\
\int_0^\alpha d\theta \tan\theta &= -\ln(\cos\alpha).
\end{aligned}$$

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1. A projectile of mass m feels a drag force,

$$|F_d| = Av^2.$$

(a) (5 pts) If it is dropped, with zero initial velocity, from a large height, what is its maximum (critical) speed v_c? Give your answer in terms of A, m and the acceleration of gravity g.

(b) (10 pts) If it is fired upward, with an initial velocity v_0, what is the time that passes before it reaches its maximum height?

(c) (10 pts) Solve for the speed as a function of time.

(d) (15 pts, extra credit, all or nothing) What is the maximum height attained by the projectile? Show that your answer gives the correct answer in the limits that the initial velocity is zero, and in the limit that $A = 0$.

(a) $mg = Av_c^2$, $V_c = \sqrt{mg/A}$

(b) $$\frac{dv}{dt} = -g - \frac{A}{m}v^2$$

$$t = -\int_{v_0}^{0} \frac{dv}{g + \frac{A}{m}v^2} = \frac{1}{g}\int_0^{v_0} \frac{dv}{1 + \frac{v^2}{v_c^2}}$$

$$= \frac{v_c}{g} \tan^{-1}\left(v_0/v_c\right)$$

(c) $$t = \frac{1}{g}\int_{v}^{v_0} \frac{dv'}{1 + v'^2/v_c^2}$$

$$= \frac{v_c}{g}\left\{\tan^{-1} v_0/v_c - \tan^{-1} v/v_c\right\}$$

$$v_c \tan\left(-\frac{gt}{v_c} + \tan^{-1}(v_0/c)\right) = V$$

$$v_c \tan\left(-\frac{gt}{v_c} + \tan^{-1}(v_0/c)\right) = v$$

$$h = \int v\,dt = v_c \int dt \tan\left(\varphi - \frac{gt}{v_c}\right)$$

$$= \frac{v_c^2}{g} \ln \cos\left(\varphi - gt/v_c\right)\Big|_0^{t_{max}}$$

$$t_{max} = \frac{v_c}{g} \tan^{-1}(v_0/v_c)$$

$$\varphi = \tan^{-1}(v_0/v_c)$$

$$h = \frac{v_c^2}{g} \ln\left[\cancel{\cos\left(\varphi - \tan^{-1}\frac{v_0}{c}\right)}\right]$$

$$- \frac{v_c^2}{g} \ln\left[\cos(\varphi)\right]$$

$$= -\frac{v_c^2}{g} \ln\left[\cos\left(\tan^{-1}\frac{v_0}{v_c}\right)\right]$$

$$= \frac{v_c^2}{2g} \ln\left(1 + \frac{v_0^2}{v_c^2}\right)$$

As $A \to 0$, $v_c \to \infty$, $\qquad (\ln(1+x) \sim x)$

$$h \to \frac{v_c^2}{2g} \frac{v_0^2}{v_c^2} = \frac{v_0^2}{2g} \quad \checkmark$$

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2. Nancy has an iceboat of initial mass M_0 which glides on a frictionless lake and has initial speed v_0. As she glides she picks up small penguins of mass m_p, who jump straight up then land on the boat as it passes by. There are ρ penguins per distance.

   (a) (10 pts) What is Nancy's speed as a function of the distance traveled x?

   (b) (10 pts) What is Nancy's position as a function of time?

   (c) (5 pts) What is Nancy's speed as a function of time?

a) $$P_{boat} = (M_0 + m_p \rho \cdot X) \cdot v = M_0 v_0$$

$$v = \frac{M_0}{M_0 + m_p \rho X} \cdot v_0$$

b) $$\frac{dx}{dt} = \frac{M_0 v_0}{M_0 + m_p \rho X}$$

$$M_0 v_0 t = \int_0^x dx' \, (M_0 + m_p \rho x')$$

$$= M_0 X + m_p \rho x^2/2$$

$$\frac{m_p \rho}{2} x^2 + M_0 X - M_0 v_0 t = 0$$

$$X = \frac{-M_0 + \sqrt{M_0^2 + 2 m_p \rho M_0 v_0 t}}{m_p \rho}$$

c) $$v = \frac{dx}{dt} = \frac{1}{m_p \rho} \cdot \frac{m_p \rho M_0 v_0}{(M_0^2 + 2 m_p \rho M_0 v_0 t)^{1/2}}$$

$$= \frac{M_0 v_0}{(M_0^2 + 2 m_p \rho M_0 v_0 t)^{1/2}}$$