your name______________________________________

*Physics 321 Exam #1 - Friday, Oct. 12, 2018*

FYI:
Some integrals:

$$\begin{aligned}
\int_0^x \frac{dy}{\sqrt{1-y^2}} &= \sin^{-1}(x), \\
\int_0^x \frac{dy}{\sqrt{1+y^2}} &= \sinh^{-1}(x), \\
\int_0^x \frac{dy}{1+y^2} &= \tan^{-1}(x), \\
\int_0^x \frac{dy}{1-y^2} &= \tanh^{-1}(x).
\end{aligned}$$

For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\
x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)} \\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

---

your name__________________________________

1. A particle of mass $m$ is in a harmonic oscillator of fundamental frequency $\sqrt{k/m} = \omega_0$, and feels a damping force $-2m\beta v$. Additionally, there is an external force,

$$F(t) = F_0\delta(t),$$

where $\delta(t)$ is a delta function. The particle is initially ($t < 0$) at rest and at the origin.

(a) (10 pts) If the motion is critically damped, find $x(t)$

(b) (15 pts) At what times does the particle cross the origin if the motion is:

- under-damped
- critically damped
- over-damped

Do not count the initial position of $x = 0$ as one of the crossings.

(a) $x(t) = Ae^{-\beta t} + Bte^{-\beta t}$

$x(0) = 0$

$\dot{x}(0) = F_0/m$

$A = 0$

$(F_0/m) = B$

$x = \frac{F_0}{m} t e^{-\beta t}$

(b) under damped $x = B \sin \omega' t \, e^{-\beta t}$

$\omega' t = n\pi$, $t = n\pi / \sqrt{\omega_0^2 - \beta^2}$

critically damped

$x = 0$ at $t = \infty$, so never

(c) $x = Ae^{-\beta_1 t} + Be^{-\beta_2 t}$

$(F_0/m) = -\beta_1 A - \beta_2 B$

$0 = A + B$

$A = \frac{(F_0/m)}{\beta_2 - \beta_1}$ $B = \frac{F_0/m}{(\beta_1 - \beta_2)} = -A$

$$x = Ae^{-\beta_1 t} - Ae^{-\beta_2 t}$$

Let $\beta_2 > \beta_1$ then $A > 0$

so $x$ is always $> 0$

never crosses.

2. A particle of mass $m$ is in a harmonic oscillator of fundamental frequency $\sqrt{k/m} = \omega_0$, and feels a damping force, $-2m\beta v$. Additionally, there is an external periodic force,

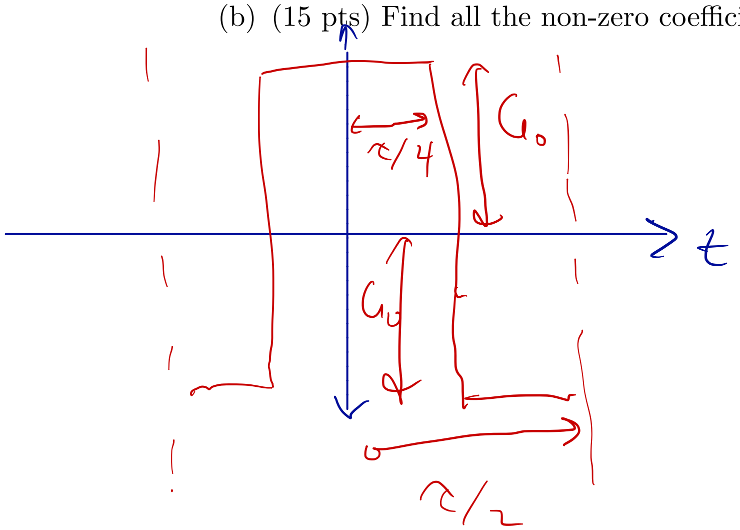
$$\begin{aligned} F(t) &= mG(t), \\ G(t) &= \begin{cases} -G_0, & -\tau/2 < t < -\tau/4 \\ +G_0, & -\tau/4 < t < \tau < 4 \\ -G_0, & \tau/4 < t < \tau/2. \end{cases} \\ G(t+\tau) &= G(t). \end{aligned}$$

(a) (10 pts) For the expansion,

$$\begin{aligned} G(t) &= \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t), \\ \omega &\equiv \frac{2\pi}{\tau}, \end{aligned}$$

For what values of $n$ are $f_n$ zero? – and for what values are $g_n$ zero?

(b) (15 pts) Find all the non-zero coefficients.




a) $g_n = 0$ for all $n$

$f_n = 0$ when $m\pi = \omega n \tau/4 = \frac{2\pi n}{\tau}\frac{\tau}{4} = \frac{\pi n}{2}$

where $m$ = some integer

$n = 0\,2, 4, 6, 8 \ldots$

b) $f_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} G\,,\ \cos 2\pi n t/\tau \, dt$, $n = 1, 3, 5, 7$

$= \frac{8G_0}{\tau}\int_0^{\tau/4} dt \cos\frac{2\pi n t}{\tau}$

$= \frac{4G}{\pi n}\sin\left(\frac{n\pi}{2}\right) = \frac{4G}{\pi n}(-1)^{2n+1}$, $n = 1, 3, 5, 7$