

Physics 321 Midterm #2 - Wednesday, Nov. 19

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$x = A_1 e^{-\beta t} \cos \omega' t + A_2 e^{-\beta t} \sin \omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad (\text{under damped})$$

$$x = A e^{-\beta t} + B t e^{-\beta t}, \quad (\text{critically damped})$$

$$x = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad (\text{over damped}).$$

Coriolis and centrifugal forces

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_{\text{real}} - m \vec{\omega} \times \vec{\omega} \times \vec{r} - 2m \vec{\omega} \times \vec{v}.$$

1. A small particle of mass m is aimed at a heavy target. The **REPULSIVE** Coulomb potential between the particles is

$$V(r) = \frac{\alpha}{r}.$$

- (a) (2 pts) If a collision occurs when the particles are separated by R or less, what is the minimum energy required for a collision?
- (b) (3 pts) If the impact parameter is b and the initial kinetic energy is E , what is the closest distance r_{min} reached during the trajectory?
- (c) (5 pts) Again assuming a collision occurs for if the particles come within R of one another, if the incoming energy of the particle is E what is the cross section σ for colliding with the target?

Solution:

a)

$$E_{\text{min}} = \frac{\alpha}{R}.$$

b)

$$E = \frac{L^2}{2mr_{\text{min}}^2} + \frac{\alpha}{r_{\text{min}}},$$

$$L^2 = 2mEb^2,$$

$$E = \frac{Eb^2}{r_{\text{min}}^2} + \frac{\alpha}{r_{\text{min}}},$$

$$Er_{\text{min}}^2 - \alpha r_{\text{min}} - Eb^2 = 0,$$

$$r_{\text{min}} = \frac{\alpha + \sqrt{\alpha^2 + 4E^2 b^2}}{2E}.$$

c)

$$ER^2 - \alpha R - Eb^2 = 0,$$

$$\pi b^2 = \pi R^2 \left(1 - \frac{\alpha}{RE} \right).$$

your name _____

2. (5 pts) A projectile is fired in Minneapolis (latitude= 45°) from a nearly horizontally aimed canon with a muzzle velocity v_0 . If the cannon is initially aimed NORTH, and if the projectile travels a distance L , what is the deflection in the EAST-WEST direction due to the Coriolis force. Assume the deflection is much smaller than L , and specify whether the deflection is in the east or west direction. Refer to Earth's angular velocity as ω_0 .

Solution:

Let \hat{x} point east, \hat{y} point north and \hat{z} point up. Earth's rotational velocity is

$$\vec{\omega} = \omega_0 \frac{\hat{z} + \hat{y}}{\sqrt{2}}.$$

The Coriolis force in the x direction is then due to ω_z and the velocity, which is then the y direction.

$$F_x = 2m \frac{\omega_0}{\sqrt{2}} v_0$$

The displacement in the x direction is then

$$\begin{aligned} \delta x &= \frac{1}{2} a_x t^2 = \frac{\sqrt{2}}{2} \omega_0 v_0 t^2, \\ t &= \frac{L}{v}, \\ \delta x &= \frac{\omega_0 L^2}{v_0 \sqrt{2}}. \end{aligned}$$

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3. After being dropped with zero initial velocity, a solid copper ball of mass m falls with a drag force γAv^2 , where A is the cross sectional area. The magnitude of the gravitational acceleration is g
- (a) (3 pts) Solve for the speed as a function of time.
 - (b) (2 pts) If two solid copper balls A and B are dropped simultaneously, one with $R_B > R_A$, which ball falls more quickly? A or B ? Explain your reasoning.

Solution:

a) Let down be the positive direction.

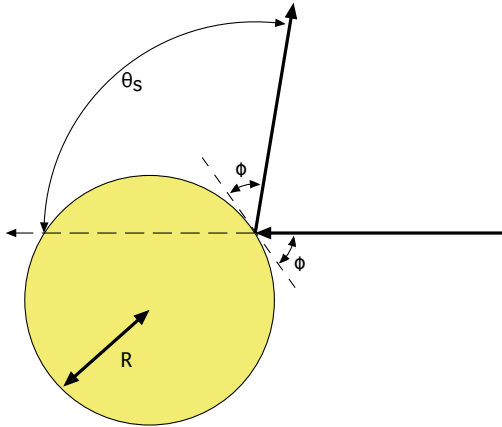
$$\begin{aligned}\frac{dv}{dt} &= g - (\gamma A/m)v^2, \\ t &= \int_0^v dv' \frac{1}{g - \gamma A f v'^2/m}, \\ &= \frac{v_0}{g} \int_0^{v/v_0} \frac{du}{1 - u^2} \\ &= \frac{v_0}{g} \tanh^{-1}(v/v_0), \quad v_0^2 \equiv mg/\gamma A\end{aligned}$$

$$v = v_0 \tanh(gt/v_0).$$

Note that the maximum velocity is v_0 , which is the same velocity one would find if one solved for zero acceleration.

b) The ball with the higher $v_0^2 = mg/\gamma A$ will move more quickly. Since the mass goes as R^3 and the area as R^2 , the larger ball will go more quickly. In other words, the gravitational force goes as R^3 and the drag as R^2 .

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4. (5 pts extra credit - all or nothing). A point particle is fired at a spherical target of radius R . The particle bounces off the target elastically with scattering angle θ_s . The angle ϕ in the figure is only meant to show that for a plane tangent to the surface, the angles relative to the surface are equal for the incoming and outgoing trajectories. Find the differential cross section $d\sigma/d\cos\theta_s$.

Solution:

An angle θ from the center of the target to the point of contact bisects the trajectory. The impact parameter is then

$$b = R \sin \theta,$$

and the scattering angle is

$$\theta_s = \pi - 2\theta.$$

Now, calculate the differential cross section

$$\begin{aligned} d\sigma &= 2\pi b db = 2\pi R^2 \sin \theta \cos \theta d\theta \\ &= \pi R^2 \sin 2\theta d\theta \\ &= \pi R^2 \sin(\pi - \theta_s) = \pi R^2 \sin \theta_s d\theta = \frac{\pi R^2}{2} \sin \theta_s d\theta_s \\ &= \frac{\pi R^2}{2} d\cos \theta_s, \\ \frac{d\sigma}{d\cos \theta_s} &= \frac{\pi R^2}{2}. \end{aligned}$$