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Physics 321 Midterm #2 - Friday, Nov. 20

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\ x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)} \\ x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).} \end{aligned}$$

Coriolis and centrifugal forces

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\rm real} - m\vec{\omega}\times\vec{\omega}\times\vec{r} - 2m\vec{\omega}\times\vec{v}.$$

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1. A small asteroid of mass m is aimed at a heavy planet of mass M and radius R. If the asteroid's speed relative to the planet is v_0 when the asteroid is far away,
   (a) (2 pts) What is the speed with which an asteroid with impact parameter $b = 0$ strikes the surface?
   (b) (2 pts) Find the maximum angular momentum that would lead to a collision given v_0?.
   (c) (1 pt) What is the maximum impact parameter, $b_{\rm max}$, that would lead to a collision?
   (d) (1 pt) What is the cross section for collision?
   (e) (1 pts) What is the maximum speed of an asteroid whose trajectory just misses colliding with the planet?

Solution: a)

$$\begin{aligned} E &= \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \frac{GMm}{r}, \\ v_{\rm max}(b=0) &= \sqrt{v_0^2 + 2\frac{GM}{R}}. \end{aligned}$$

b)

$$\begin{aligned} \frac{L^2}{2mR^2} - \frac{GMm}{R} &= \frac{1}{2}mv_0^2, \\ L &= mR\sqrt{\frac{2GM}{R} + v_0^2}, \end{aligned}$$

c)

$$b = L/(mv_0) = R\sqrt{1 + \frac{2GM}{Rv_0^2}}.$$

d)

$$\sigma = \pi b_{\rm max}^2 = \pi R^2 \left(1 + \frac{2GM}{Rv_0^2}\right).$$

e)

$$v_{\rm max}(b_{\rm max}) = \frac{L}{mR} = \sqrt{\frac{2GM}{R} + v_0^2}$$

could also use energy conservation, same as (a).

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2. A ball of mass m is dropped from a height, h, above Minneapolis (latitude=45°). It falls without air resistance.

   (a) (4 pts) How far is the ball deflected by the Coriolis force? Give answer in terms of m, g, h and Earth's radius R and period T.

   (b) (1 pt) In what direction is the deflection? (e.g. southwest)

Solutions:

a) Let $\hat{x}$ point east, $\hat{y}$ point north and $\hat{z}$ point up.

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\begin{aligned}
F_x &= -2m\omega_y v_z, \\
F_y &= 2m\omega_x v_z, \\
\omega_y &= \frac{2\pi}{T\sqrt{2}}, \quad \omega_x = 0, \\
F_y &= 0, \quad F_x = -2m\frac{2\pi}{T\sqrt{2}}v_z. \\
v_z &= -gt, \\
a_x &= \frac{2\pi g\sqrt{2}}{T}t, \\
v_x &= \frac{\pi g\sqrt{2}}{2T}t^2, \\
x &= \frac{\pi g\sqrt{2}}{6T}t^3, \\
\frac{1}{2}gt^2 &= h, \\
t &= \sqrt{2h/g} \\
x &= \frac{\pi g\sqrt{2}}{3T}\frac{(2h)^{3/2}}{2g^{3/2}} \\
&= \frac{2\pi h^{3/2}}{3Tg^{1/2}}.
\end{aligned}
$$

b) east

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3. A particle of mass m is in a circular orbit or radius R according to a potential

$$V = V_0 \ln(r/a).$$

(a) (1 pt) What is the particle's speed while in a circular orbit? Give answer in terms of V_0, a, m and R.

(b) (1 pt) What is the particle's angular momentum, L, in the circular orbit? Give answer in terms of V_0, a, m and R.

(c) (2 pts) For a particle with this angular momentum, what is the effective radial potential? Provide a sketch. Label the radius of a circular orbit.

(d) (3 pts) What is the angular frequency of small oscillations of the radial distance r about a circular orbit? Give answer in terms of V_0, a, m and R.

Solutions: a)

$$\begin{aligned} F &= -\frac{V_0}{r}, \\ &= -m\frac{v^2}{r}, \\ v &= \sqrt{V_0/m}. \end{aligned}$$

b)

$$L = mvR = R\sqrt{mV_0}$$

c)

$$\begin{aligned} V_{\rm eff} &= \frac{L^2}{2mr^2} + V_0 \ln(r/a), \\ 0 &= \frac{dV}{dr} = -\frac{L^2}{mr^3} + \frac{V_0}{r}, \\ L^2 &= mV_0R^2, \\ k_{\rm eff} &= \frac{d^2V}{dr^2} = \frac{3L^2}{mR^4} - \frac{V_0}{R^2}, \\ &= \frac{3V_0R^2}{R^4} - \frac{V_0}{R^2} \\ &= \frac{2V_0}{R^2}, \\ \omega &= \sqrt{k_{\rm eff}/m} = \frac{\sqrt{2V_0/m}}{R}. \end{aligned}$$

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4. A particle of mass m moves according to the potential

$$V(x) = \alpha x, \quad \alpha > 0.$$

A particle is sent in from the left (from $x < 0$), from position $-a$ with total energy $E > 0$ (potential plus kinetic).

(a) (1 pt) Sketch the potential, $V(x)$. Mark the energy E on the potential axis, and label the point at which the particle turns around.

(b) (1 pts) If the particle's initial velocity is toward the origin, what is the maximum position, $x_{\rm max}$, of its trajectory?

(c) (3 pts) How much time is required to move to that maximum position?

Solution:

a) Hopefully you can sketch this

b)E/α.

c)

$$\begin{aligned}
t &= \int_{-a}^{x} \frac{dx'}{v} \\
&= \int_{-a}^{x} \frac{dx'}{\sqrt{2(E-\alpha x)/m}} \\
&= \sqrt{\frac{m}{2\alpha}} \int_{-a}^{x} \frac{dx'}{\sqrt{(E/\alpha) - x}} \\
&= -\sqrt{\frac{2m}{\alpha}} \sqrt{(E/\alpha) - x'}\Big|_{-a}^{x} \\
&= \sqrt{\frac{2m}{\alpha}} \left(\sqrt{(E/\alpha) + a} - \sqrt{(E/\alpha) - x} \right) \\
&= \sqrt{(2mE/\alpha^2) + 2ma/\alpha}
\end{aligned}$$