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Physics 321 FINAL - Thursday, Dec. 17 2015

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)}\\
x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).}
\end{aligned}$$

Coriolis and centrifugal forces

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\rm real} - m\vec{\omega}\times\vec{\omega}\times\vec{r} - 2\vec{\omega}\times\vec{v}.$$

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}} = \frac{\partial L}{\partial q}.$$

Some integrals:

$$\begin{aligned}
\int\frac{dx}{1-x^2} &= \tanh^{-1}(x),\\
\int\frac{dx}{1+x^2} &= \tan^{-1}(x),\\
\int\frac{dx}{\sqrt{x^2-1}} &= \cosh^{-1}(x),\\
\int\frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x),\\
\int\frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x),\\
\int_{-\infty}^{\infty} dx\;\delta(x) &= 1.
\end{aligned}$$

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1. A projectile of mass m feels a gravitational force mg, plus a drag force, γv^2.

   (a) (1pt) What is the terminal velocity, v_t, of a ball dropped from a large height?

   (b) (3 pts) If the projectile is hurled UPWARD with initial speed v_0, find the speed as a function of time on the way up.

   (c) (1 pt) Regarding the upward trajectory above, how much time passes before the projectile reaches its maximum height?

Solution:

a)

$$\begin{aligned} mg &= \gamma v_t^2, \\ v_t &= \sqrt{mg/\gamma}. \end{aligned}$$

b)

$$\begin{aligned} m\frac{dv}{dt} &= -\gamma v^2 - mg, \\ t &= -\int \frac{dv}{g + \gamma v^2/m}, \\ gt &= -\int \frac{dv}{1 + v^2/v_t^2}, \\ \frac{gt}{v_t} &= -\int_{v_0/v_t}^{v/v_t} \frac{dx}{1+x^2} = -\tan^{-1}(v/v_t) + \tan^{-1}(v_0/v_t), \\ v &= v_t \tan\left(\tan^{-1}(v_0/v_t) - \frac{gt}{v_t}\right) \end{aligned}$$

c)

$$t_{\max} = \frac{v_t}{g}\tan^{-1}\left(\frac{v_0}{v_t}\right).$$

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2. (5 pts) A particle of mass m moving in one dimension is confined by a spring with spring constant k, and also experiences a small dissipative force $-bv$. For negative times, the particle is at rest at the equilibrium position $x = 0$. The particle then feels a sudden impulse I, i.e. the force is of the form $F = I\delta(t)$. Find the position as a function of time.

Solution:

$$\begin{aligned}
mv_0 &= I, \quad v_0 = I/m, \\
x &= Ae^{-\beta(t-t_0)}\cos\omega'(t-t_0) + Be^{-\beta(t-t_0)}\sin\omega'(t-t_0), \quad \beta \text{ and } \omega' \text{ defined above}, \\
0 &= A, \\
I/m &= \omega' B, \\
x &= \frac{I}{m\omega'}e^{-\beta(t-t_0)}\sin\omega'(t-t_0).
\end{aligned}$$

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3. (5 pts) A particle feels an attractive central potential,

$$V(r) = \beta r.$$

The particle is in a stable circular orbit of angular frequency ω_0, when it feels a small perturbation which causes the the radius r to oscillate about the original orbit's radius with frequency ω. Find ω in terms of ω_0.

Solution:

$$\begin{aligned}
V_{\rm eff} &= \beta r + \frac{L^2}{2mr^2}, \\
\frac{d^2 V_{\rm eff}}{dr^2} &= k_{\rm eff} = \frac{3L^2}{mr^4}, \\
\omega^2 &= \frac{3L^2}{m^2 r^4}, \\
L &= mr^2\omega_0, \\
\omega &= \omega_0\sqrt{3}.
\end{aligned}$$

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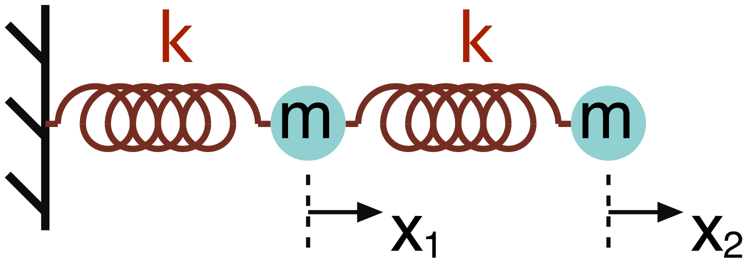

4. Consider the two identical masses connected to the two identical springs pictured above. Let x_1 and x_2 describe the displacement of the two masses relative to their fixed equilibrium position.

   (a) (3 pts) Write the Lagrangian in terms of x_1 and x_2, then find the equations of motion.

   (b) (3 pts) Assume there are solutions of the form,

   $$x_1 = Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}.$$

   Find the frequencies of the two normal modes.

Solution:

a)

$$\begin{aligned}
\mathcal{L} &= \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 - \frac{k}{2}x_1^2 - \frac{k}{2}(x_1 - x_2)^2, \\
\ddot{x}_1 &= -\omega_0^2 x_1 - \omega_0^2(x_1 - x_2) = -2\omega_0^2 x_1 + \omega_0^2 x_2, \\
\ddot{x}_2 &= -\omega_0^2(x_2 - x_1)
\end{aligned}$$

b)

$$\begin{aligned}
x_1 &= Ae^{i\omega t}, \quad x_2 = Be^{i\omega t}, \\
-\omega^2 A &= -\omega_0^2(2A - B), \\
-\omega^2 B &= -\omega_0^2(B - A), \\
-\omega^2 A &= -\omega_0^2\left(2A - \frac{\omega_0^2}{\omega_0^2 - \omega^2}A\right), \\
-\omega^2 &= -2\omega_0^2 + \frac{\omega_0^4}{\omega_0^2 - \omega^2}, \\
\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 &= 0, \\
\omega &= \omega_0\sqrt{\frac{3 \pm \sqrt{5}}{2}}
\end{aligned}$$

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5. Consider a particle of mass m confined to move along the surface of a cone defined by

$$z = \alpha\rho, \quad \rho = \sqrt{x^2 + y^2}, \quad \alpha = 1.$$

The particle moves without friction along the surface and feels the gravitational force which is directed in the negative z direction.

(a) (2 pts) Write down the Lagrangian in terms of the two coordinates ρ and θ, where θ is the azimuthal angle about the z axis.

(b) (2 pts) Of the following quantities, list all those that represent constants of the motion (conserved quantities for all trajectories): Energy E, components of the momentum p_x, p_y, p_z, magnitude of the momentum $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$, components of the angular momentum L_x, L_y, L_z, magnitude of the angular momentum $L = \sqrt{L_x^2 + L_y^2 + L_z^2}$.

Solution:

a) Use the fact that $\dot{z} = \dot{\rho}$,

$$\mathcal{L} = \frac{1}{2} m\rho^2\dot{\theta}^2 + m\dot{\rho}^2 - mg\rho. \tag{1}$$

b) E and L_z.