your name____

Physics 321 Final Exam - Tuesday, Dec. 11, 2018

A perfect score on this exam is 140 points. However, the exam is graded on a scale of 120 so the last 20 points can be considered as extra credit.
FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\ x &= A e^{-\beta t} + B t e^{-\beta t}, \quad \text{(critically damped)} \\ x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped)}. \end{aligned}$$

Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}.$$

Rotating frame:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{real}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}.$$

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Final Exam - Wednesday, May 2, 2018

1. Consider a particle of mass m in a one-dimensional harmonic oscillator of spring constant k and with an extremely small damping force $-bv$, i.e. $b \to 0$. Additionally, there is an external force

$$F(t) = \begin{cases} -F_0, & -\tau/2 < t < 0, \\ +F_0, & 0 < t < \tau/2 \end{cases}$$
$$F(t+\tau) = F(t).$$

Consider the expansion

$$F(t) = a_0 + \sum_{n>0} a_n \cos(n\omega t) + b_n \sin(n\omega t),$$

(a) (5 pts) What is ω?

(b) (5 pts) For what n are the coefficients $a_n = 0$ and for what n are $b_n = 0$?

(c) (10 pts) NOT SOLVING FOR a_n and b_n, express $x(t)$ in terms of a_n and/or b_n assuming the force has existed for a long time ("long time" means that the damping has eliminated all transients even though the damping is extremely small). Your answer should be of the form

$$x(t) = \sum_n \cdots,$$

(The damping b should not appear in your answer)

a) $\omega = 2\pi/\tau$

b) $a_n = 0$ for all n

$b_n = 0$ for $n = 2, 4, 5, 6$

c) $X(t) = \sum_{n=1,3,5} \beta_n \sin(n\omega t)$

$\ddot{X} + \omega_0^2 X = \frac{1}{m} \sum_n b_n \sin(n\omega t)$

$(-n^2\omega^2 + \omega_0^2)\beta_n = \frac{1}{m} b_n$

$\beta_n = \frac{b_n}{m} \frac{F}{(-n^2\omega^2 + \omega_0^2)}$

$x(t) = \sum_n \frac{b_n}{m} \frac{1}{(-n^2\omega^2 + \omega_0^2)} \sin(n\omega t)$

Extra work space for #1

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2. A ball of mass m is dropped (initial velocity is zero) from a height h_0. The ball experiences a drag force $-bv$.

   (a) (5 pts) What is the terminal velocity?

   (b) (10 pts) What is the velocity as a function of time? (v should be negative for downward motion)

   (c) (5 pts) What is the height as a function of time?

   (d) (5 pts) A second ball is released from the same point, but initially moving downward with the terminal velocity. After a long time has passed, and both balls are moving with the ~~thermal~~ terminal velocity, what height separates the two balls?

a) $$\frac{dv}{dt} = -g - \frac{b}{m}v$$

$$v_{term} = -mg/b$$

b) $$\frac{dv}{\frac{b}{m}\left(v + \frac{mg}{b}\right)} = -dt$$

$$\ln \frac{v + v_{term}}{v_{term}} = -\frac{b}{m}t$$

$$v + v_{term} = v_{term}\, e^{-b/m\, t}$$

$$v = -v_{term} + v_{term}\, e^{-\frac{b}{m}t}$$

c) $$x = -v_{term}\,t - v_{term}\cdot\frac{m}{b}\left(e^{-\frac{b}{m}t} - 1\right)$$

$$= -v_{term}\,t + \frac{m^2 g}{b^2}\left(1 - e^{-\frac{b}{m}t}\right)$$

d) $$x_2 = -v_{term}\,t$$

$$\Delta x = \frac{m^2 g}{b^2} = \frac{v_{term}^2}{g}$$

Extra work space for #2

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3. A particle of mass m is aimed at a spherical target of radius R. The particle feels a repulsive potential from the target

$$V(r) = \frac{\alpha}{r},$$

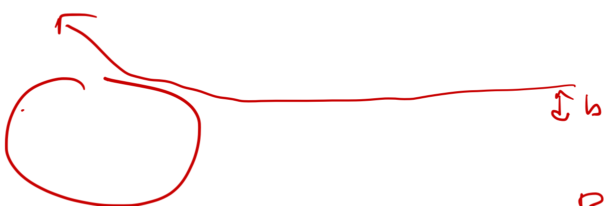
where r is the distance to the particle from the center of the target sphere.

(a) (5 pts) What is the minimum energy the particle must have in order to impact the surface of the sphere?

(b) (15 pts) Assuming the energy exceeds this threshold, what is the cross section for collision with the sphere?
Give your answer in terms of R, α, m and the energy E.

a) $E_{min} = \alpha / R$

b)



$$L = m v_0 b = m v_f R$$

$$m \frac{v_f^2}{2} = \frac{m v_0^2}{2} - \frac{\alpha}{R}$$

$$v_f = v_0 \sqrt{1 - \frac{\alpha}{R} \frac{2}{m v_0^2}}$$

$$b = R \cdot \sqrt{1 - \frac{\alpha}{R} \frac{1}{E}}$$

$$\sigma = \pi R^2 \left(1 - \frac{\alpha}{ER}\right)$$

Extra work space for #3

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4. Consider two equal masses m confined to moving in the $z = 0$ plane. The masses are connected by a massless spring of spring-constant k whose relaxed length is ℓ. In terms of the coordinates of the two particles, the potential energy of the spring is

$$U = \frac{k}{2}\left[\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - \ell\right]^2 .$$

For generalized coordinates, use the center-of-mass coordinates, $X = (x_1 + x_2)/2$, $Y = (y_1 + y_2)/2$, the distance between the masses $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and the angle of the spring relative to the x axis, ϕ.

(a) (10 pts) Write the Lagrangian in terms of $X, Y, r, \phi, \dot{X}, \dot{Y}, \dot{r}$ and $\dot{\phi}$.

(b) (10 pts) Write the four equations of motion.

(c) (10 pts) Write as many conserved quantities as you can think of in terms of $m, k, \ell, X, Y, r, \phi, \dot{X}, \dot{Y}, \dot{r}$ and $\dot{\phi}$.

a) $T = T_{cm} + T_{rel} = \frac{1}{2} 2m (\dot{X}^2 + \dot{Y}^2) + \frac{\mu}{2}\dot{r}^2 + \frac{2}{2} m \frac{r^2}{4}\dot{\varphi}^2$

$= m(\dot{X}^2 + \dot{Y}^2) + m\frac{\dot{r}^2}{4} + m\frac{r^2}{4}\dot{\varphi}^2$

$L = m(\dot{X}^2 + \dot{Y}^2) + \frac{m}{4}\dot{r}^2 + \frac{m r^2}{4}\dot{\varphi}^2 - \frac{k}{2}(r - \ell)^2$

b) $\ddot{X} = 0$

$\ddot{Y} = 0$

$\frac{d}{dt}(r^2\dot{\varphi}) = 0$

$\frac{m}{2}\ddot{r} = \frac{m}{2} r \dot{\varphi}^2 - k(r - \ell)$

c) $E,\ r^2\dot{\varphi},\ \dot{X},\ \dot{Y}$

Extra work space for #4

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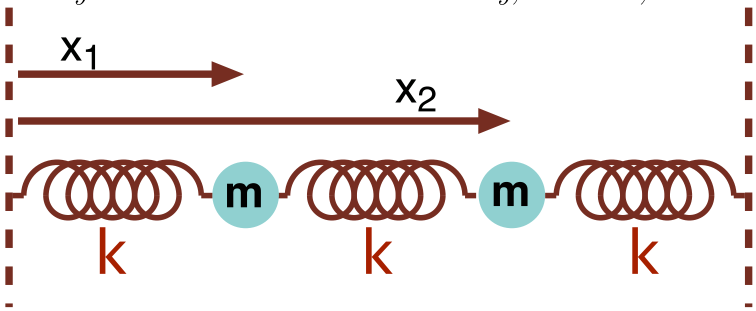

5. Consider the two masses m connected to the three identical massless springs, each with spring constant k, that are fixed at the far ends (dashed lines). At equilibrium, each spring is at its relaxed length ℓ.

   (a) (10 pts) Write the Lagrangian using the coordinates x_1 and x_2.

   (b) (5 pts) Write the equations of motion for x_1 and x_2.

   (c) (10 pts) Describe the two normal modes and their frequencies.

a) $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

$U = \frac{1}{2} k (x_1 - \ell)^2 + \frac{1}{2} k (x_2 - x_1 - \ell)^2 + \frac{1}{2} k (x_2 - 2\ell)^2$

$= \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k x_2^2$

$- k\ell x_1 + k(x_1 - x_2)\ell - 2k x_2 \ell$

$= k x_1^2 + k x_2^2 - k x_1 x_2 - 3k x_2 \ell + 3k\ell^2$

$\mathcal{L} = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - k x_1^2 - k x_2^2 + k x_1 x_2 - 3k(x_2 - \ell)\ell$

b) $m \ddot{x}_1 = -2k x_1 + k x_2$

$m \ddot{x}_2 = -2k x_2 + k x_1 - 3k\ell$

c) $x_1 = \ell + A e^{i\omega t}$, $x_2 = 2\ell + B e^{i\omega t}$

$-m\omega^2 A = -2kA + kB$

$-m\omega^2 B = -2kB + kA$

$\frac{A}{B} = 2 - \frac{m\omega^2}{k}$

Extra work space for #5

$$-m\omega^2 A = -2kA + kB$$

$$-m\omega^2 B = -2kB + kA$$

$$\frac{A}{B} = 2 - \frac{m\omega^2}{k}$$

$$-m\omega^2\left(2 - \frac{m\omega^2}{k}\right) = -2k\left(2 - \frac{m\omega^2}{k}\right) + k$$

$$\frac{m^2}{k}\omega^4 - 4m\omega^2 + 3k = 0$$

$$\omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4 = 0, \quad \omega_0^2 = k/m$$

$$\omega^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 12\omega_0^4}}{2}$$

$$= 2\omega_0^2 \pm \omega_0^2 = 3\omega_0^2, \omega_0^2$$

$$\frac{A}{B} = 2 - \frac{\omega^2}{\omega_0^2} = -1, 1$$

Mode #1, $x_1 = \ell - B\cos(\sqrt{3}\,\omega_0 t)$

$x_2 = 2\ell + B\cos(\sqrt{3}\,\omega_0 t)$

Mode #2, $x_1 = \ell + B\cos(\omega_0 t + \varphi)$

$x_2 = 2\ell + B\cos(\omega_0 t + \varphi)$

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6. A mass m is dropped (initial velocity is zero) from a large height (but not so large you can't consider g constant) above a point on the equator. Neglect air resistance. Let Earth's ~~rotational~~ angular velocity $= \omega$

   (a) (15 pts) Relative to dropping exactly below where it was dropped, find the distance to where the mass hits the ground.

   (b) (5 pts) In which direction (north, south, east or west) is this position relative to directly below the dropping point.

$\hat{x}$ = EAST

$$m\frac{d\vec{v}}{dt} = \vec{F}_{\text{real}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}.$$

$$m\frac{dv_x}{dt} = -2m\omega|v_z|\,\hat{y} \times (-\hat{z})$$

$$= 2m\omega(gt)$$

$$\frac{dv_x}{dt} = 2\omega g t$$

$$v_x = \omega g t^2$$

$$x = \frac{1}{3}\omega g t^3$$

$$h = \frac{1}{2} g t^2$$

$$t = (2h/g)^{1/2}$$

$$x = \frac{1}{3}\omega \frac{(2h)^{3/2}}{g^{1/2}}$$

b) EAST

Extra work space for #6