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Final Exam - Wednesday, May 1st 2019

This exam has 8 problems, with 130 total possible points. However, the exam will be graded on a scale of 120, so a perfect performance earns 130/120.
FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned}
x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \qquad \text{(under damped)}\\
x &= A e^{-\beta t} + B t e^{-\beta t}, \qquad \text{(critically damped)}\\
x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped)}.
\end{aligned}$$

Fourier expansion ($\omega = 2\pi/\tau$):

$$\begin{aligned}
F(t) &= \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t),\\
f_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\cos(n\omega t),\\
g_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} dt\ F(t)\sin(n\omega t).
\end{aligned}$$

Some integrals:

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \tan^{-1}(x),\\
\int \frac{dx}{1-x^2} &= \tanh^{-1}(x),\\
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1}(x),\\
\int \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}(x),
\end{aligned}$$

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1. (15 pts) You are designing a remote-controlled ice-boat to move along a frictionless lake. You can use two boats, attached to each other, and each boat has a machine gun. The mass of each boat with its machine gun included is m_g. Each boat has the capacity to carry a mass of bullets m_b. The boats accelerate by firing the machine gun. You can unattach the boats after boat #1 has expended its ammunition, then further accelerate boat #2 by firing its ammunition. What is the maximum speed boat #2 can attain?

bullet speed is v_b

v_1 = speed gained in first stage

v_2 = " " second stage

$$v_1 = v_b \ln \frac{2m_b + 2m_g}{2m_g + m_b}$$

$$v_2 = v_b \ln \frac{m_b + m_g}{m_g}$$

$$v_{max} = v_b \ln \left\{ \frac{2m_g + 2m_b}{2m_g + m_b} \frac{m_b + m_g}{m_g} \right\}$$

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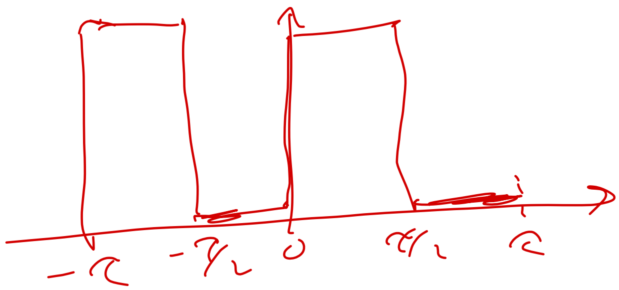
2. Consider the time-varying potential that is periodic in time, $V(t+\tau) = V(t)$.

$$V(t) = \begin{cases} 0, & -\tau/2 < t < 0 \\ V_0, & 0 < t < \tau/2 \end{cases}$$

Let the potential be expressed as a Fourier series,

$$V(t) = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos(n\omega_0 t) + g_n \sin(n\omega_0 t),$$
$$\omega_0 \equiv 2\pi/\tau.$$

(a) (5 pts) Graph the potential for the interval $-\tau < t < \tau$.



(b) (5 pts) For which values of n are $f_n \neq 0$?

$f_0 \neq 0$

(c) (5 pts) For which values of n are $g_n \neq 0$?

1, 3, 5, 7

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3. According to Big Bob's theory, the cross section for a particle with kinetic energy E to scatter by a scattering angle θ_s off a gold target, is:

$$\frac{d\sigma}{d\Omega} = \alpha(1 + \beta\cos\theta_s).$$

You are an experimentalist and want to measure the differential cross section $d\sigma/d\Omega$. You have an array of detectors at numerous angles. Each detector element has a facial area a and the detector elements are located a distance L from the target, with $a/L^2 << 1$. The efficiency of each detector element is f. The target is a thin foil of gold, whose thickness is t. The mass of a gold atom is m_{Au}, and the mass density of gold is ρ_{Au}. Give answers in terms of ρ_{Au}, m_{Au}, t, α, β, f, a and L.

(a) (5 pts) What is the total cross section?

(b) (5 pts) What fraction of beam particles scatter?

(c) (10 pts) To test Big Bob's theory, you wish to run an experiment long enough so that a single detector element at $\theta_s = 90$ degrees would record 1000 counts if Big Bob is correct. You need to tell the beam operator how many beam particles, N_b, you need to have incident on the target. What is N_b?.

a) $\sigma = 4\pi\alpha$

b) $f_{scatt} = \sigma \cdot \frac{dN_{target}}{dA} = \sigma \rho_{Au} \frac{t}{m_{Au}}$

c) $N_{count} = f \cdot \left(\frac{d\sigma}{d\Omega}\Delta\Omega\right) \cdot \rho_{Au} \frac{t}{m_{Au}} \cdot N_b$

$= \alpha f \frac{a}{L^2} \rho_{Au} \frac{t}{m_{Au}} \cdot N_b$

$N_b = \dfrac{1000 \cdot L^2 m_{Au}}{\alpha f a \rho_{Au} t}$

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4. **A particle of mass m moves according to an attractive potential**

$$V(r) = -\frac{\beta}{r^2}.$$

(a) **(5 pts) Are there any stable orbits?**

NO

(b) **(10 pts) A beam of particles with energy E is aimed at the potential. If the particle reaches the origin, it is annihilated. What is the cross section for annihilation? Give your answer in terms of m, E and β.**

$$\frac{L^2}{2m} = \beta = \frac{m^2 v^2 b^2}{2m} = \frac{m^2 v^2 \sigma}{2m\pi}$$

$$\sigma = \frac{\pi \beta}{E}$$

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5. A bead of mass m moves along a frictionless circular wire of radius R. The plane of the circle is vertical, so that gravity will pull it down to the $\theta = 0$.

(a) (10 pts) Using θ as the generalized coordinate, write the Lagrangian.

(b) (5 pts) What are the equations of motion for θ?

a) $\mathcal{L} = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR(1-\cos\theta)$

$$m R^2 \ddot{\theta} = -mgR\sin\theta$$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta$$

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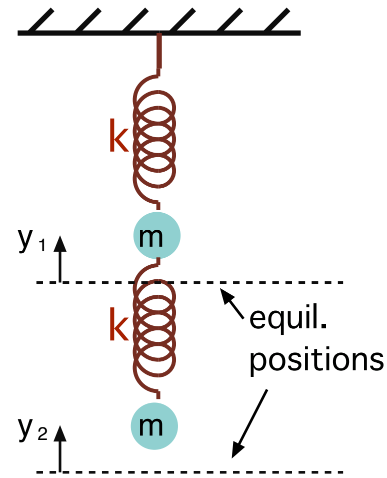

6. Two massless springs are attached to the ceiling a shown. Each spring is characterized by a spring constant k, and the two masses, m are equal. Assume the masses move only vertically. As generalized coordinates, let y_1 and y_2 refer to the displacements of the upper and lower springs respectively. These displacements are relative to fixed points, being the equilibrium positions. That is, at equilibrium $y_1 = y_2 = 0$. And if both mass move together $(y_1 = y_2)$, the lower spring does not stretch.

(a) (10 pts) Write down the Lagrangian.

(b) (5 pts) What are the equations of motion for y_1 and y_2?

(c) (15 pts) Describe the normal modes. Give both frequencies along with a description of how the two displacements are related for each mode.

a) $\mathcal{L} = \frac{1}{2} m (\dot{y}_1^2 + \dot{y}_2^2) - \frac{1}{2} k \{ y_1^2 + (y_1 - y_2)^2 \}$

b) $m \ddot{y}_1 = -k(2y_1 - y_2)$

$m \ddot{y}_2 = -k(y_2 - y_1)$

c) $y_1 = A e^{i\omega t}$, $y_2 = B e^{i\omega t}$, $\omega_0^2 \equiv k/m$

$-\omega^2 A = -\omega_0^2 (2A - B)$

$-\omega^2 B = -\omega_0^2 (B - A)$

$\frac{2A - B}{A} = \frac{B - A}{B}$, $2AB - B^2 = AB - A^2$

$A^2 + AB - B^2 = 0$

$\frac{A}{B} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\omega^2 = \omega_0^2 \left(1 - \frac{A}{B}\right) = \omega_0^2 \left\{ \frac{3}{2} \mp \frac{\sqrt{5}}{2} \right\}$

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7. (10 pts) particle moves in three dimensions under the influence of a potential

$$V(x, y, z) = V_0 e^{-(x^2+y^2+z^2)/b^2}.$$

If the total energy is E, the kinetic energy is T, the momentum is $\vec{p}$ and the angular momentum is $\vec{L}$, circle all the conserved quantities.

(a) E

(b) T

(c) p_x

(d) p_y

(e) p_z

(f) $|\vec{p}|$

(g) L_x

(h) L_y

(i) L_z

(j) $|\vec{L}|$

your name____

8. (10 pts) particle moves in three dimensions under the influence of a potential

$$V(x,y,z) = V_0 e^{-[(x+y)^2+z^2]/b^2}.$$

If the total energy is E, the kinetic energy is T, the momentum is $\vec{p}$ and the angular momentum is $\vec{L}$, circle all the conserved quantities.

(a) E

(b) T

(c) $p_x - p_y$

(d) $p_x + p_y$

(e) p_z

(f) $|\vec{p}|$

(g) $L_x - L_y$

(h) $L_x + L_y$

(i) L_z

(j) $|\vec{L}|$