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Physics 321 Practice Exam #1 - Wednesday, Oct 8

FYI: For the differential equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0,$$

the solutions are

$$\begin{aligned} x &= A_1 e^{-\beta t}\cos\omega' t + A_2 e^{-\beta t}\sin\omega' t \quad \omega' = \sqrt{\omega_0^2 - \beta^2} \quad \text{(under damped)} \\ x &= Ae^{-\beta t} + Bte^{-\beta t}, \quad \text{(critically damped)} \\ x &= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}, \quad \beta_i = \beta \pm \sqrt{\beta^2 - \omega_0^2}, \quad \text{(over damped).} \end{aligned}$$

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1. A particle of mass m moves in a potential $V(x) = \beta x$. The particle is initially at $x = 0$ with a kinetic energy T_0 moving to the right.
   (a) (1 pt) What is the maximum distance the particle travels before turning around?
   (b) (3 pts) Solve for the position as a function of time. Only worry about the times before the particle turns around.
   (c) (1 pt) How much time is required for the particle to turn around?

Solution:

a)

$$\begin{aligned} \beta x_{\rm max} &= T_0, \\ x_{\rm max} &= T_0/\beta. \end{aligned}$$

b)

$$\begin{aligned} t(x) &= \int_0^x \frac{dx'}{\sqrt{2(T_0 - \beta x')/m}} \\ &= \sqrt{\frac{m}{2\beta}} \int_0^x \frac{dx'}{\sqrt{(T_0/\beta) - x'}} \\ &= -\sqrt{\frac{2m}{\beta}} \sqrt{x_{\rm max} - x'}\Big|_0^x \\ &= \sqrt{\frac{2m x_{\rm max}}{\beta}} - \sqrt{\frac{2m(x_{\rm max} - x)}{\beta}} \\ x &= x_{\rm max} - \frac{\beta}{2m}\left(\sqrt{\frac{2m x_{\rm max}}{\beta}} - t\right)^2 \end{aligned}$$

c)

$$\begin{aligned} t(x_{\rm max}) &= \sqrt{\frac{m}{2\beta}} \int_0^{x_{\rm max}} \frac{dx}{\sqrt{x_{\rm max} - x}} \\ &= \sqrt{\frac{2m x_{\rm max}}{\beta}} = \sqrt{\frac{2mT_0}{\beta^2}} \end{aligned}$$

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2. (5 pts) A rocket is in deep space, out of the realm of gravitational fields, when it fires its engines and sends its exhaust backward at a speed of v_e relative to the rocket. The rocket has two stages of equal mass M filled with fuel, plus a payload of mass P. Both stages have a fraction f of their mass remaining after they have expelled all their fuel. Find the final speed of the payload after both stages have expended their fuel.

Solution:
For a single stage the final velocity is found by

$$\begin{aligned} M(t)\frac{dv}{dt} &= v_e\frac{dM}{dt}, \\ v_f &= v_e \ln\left(\frac{M_f}{M_0}\right). \end{aligned}$$

For the two stages

$$v = v_e \ln\left(\frac{P + M + fM}{2M + P}\right) + v_e \ln\left(\frac{P + fM}{P + M}\right) \tag{1}$$

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3. Consider an underdamped one-dimensional harmonic oscillator of a particle of mass m where the spring constant is k and the drag force is $-bv$.

   (a) (2 pts) Write the general solution in terms of two arbitrary constants.

   (b) (3 pts) Now, add an external force, $F_{\rm ext}(t) = mge^{-\lambda t}\Theta(t)$. Find a particular solution for $t > 0$ that behaves as $\sim e^{-\lambda t}$. ($\Theta(t) = 1$ for $t > 0$, =0 otherwise)

   (c) (3 pts) If at $t = 0$ the particle is at position x_0 with zero velocity, find the exact solution $x(t)$ for $t > 0$.

   (d) (2pts) Consider the limit where the force is very strong, but is applied for a short time. $F_0 \to \infty, \lambda \to \infty$ and $F_0/\lambda = I_0$. Assuming the particle is at rest at the origin, find $x(t)$ for $t > 0$. Given answer in terms of β, ω_0 and the initial impulse I_0.

Solution:

a)

$$\begin{aligned} x &= \Re A e^{i\omega t} \\ -mA\omega^2 &= -ib\omega A - kA, \\ \omega &= i\beta + \sqrt{\beta^2 + 4\omega_0^2}, \quad \beta \equiv b/2m, \ \omega_0 \equiv \sqrt{k/m} \\ x &= Ae^{-\beta t}\cos(\omega t) + Be^{-\beta t}\sin(\omega t), \quad \omega \equiv \sqrt{\beta^2 + \omega_0^2}. \end{aligned}$$

b)

$$\begin{aligned} x_p &= Ae^{-\lambda t}, \\ \left(\lambda^2 A - 2\beta\lambda A + \omega_0^2 A\right) e^{-\lambda t} &= ge^{-\lambda t}, \\ A &= \frac{g}{\lambda^2 - 2\beta\lambda + \omega_0^2}. \end{aligned}$$

c)

$$\begin{aligned} x &= A\cos(\omega t)e^{-\beta t} + B\sin(\omega t)e^{-\beta t} + \frac{ge^{-\lambda t}}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \\ x_0 &= A + \frac{g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \\ 0 &= -\beta A + \omega B - \frac{\lambda g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \\ A &= x_0 - \frac{g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \\ B &= \frac{\beta}{\omega} A + \frac{\lambda g/\omega}{\lambda^2 - 2\beta\lambda + \omega_0^2} \\ &= \frac{\beta x_0}{\omega} + \frac{g}{\omega}\frac{\beta + \lambda}{\lambda^2 - 2\beta\lambda + \omega_0^2}. \end{aligned}$$

d) The initial impulse is the same as the initial momentum, so $I_0 = F_0/\lambda$ and $v_0 = I_0/m$.

$$0 = A + \frac{g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \tag{2}$$

$$I_0/m = -\beta A + \omega B - \frac{\lambda g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, \tag{3}$$

$$
\begin{aligned}
A &= -\frac{g}{\lambda^2 - 2\beta\lambda + \omega_0^2}, && (4)\\
B &= \frac{1}{\omega}\left\{\frac{I_0}{m} + \beta A + \frac{\lambda g}{\lambda^2 - 2\beta\lambda + \omega_0^2}\right\} && (5)\\
&= \frac{1}{\omega}\left\{\frac{I_0}{m} + \frac{(\lambda - \beta) g}{\lambda^2 - 2\beta\lambda + \omega_0^2}\right\}. && (6)
\end{aligned}
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