John Buhl
Mark Landry
Robert Lemense

Trajectory from Potentials

Particles with mass 1 move parallel along the x axis into a potential V(x,y), which begins at x=0. The potential is given as:

$$V(x,y) = X(x)Y(y) = u(x)(\cos(y)+1)$$

At time t=0, the particles enter the potential. Assume the particles each have large energy, so they do not reflect off the potential.

Part 1.
Using an approximation, expand the motion of the particle in y, given in terms of the impact parameter from the nearest maxima of Y(y).
Part 2.
Using another appropriate approximation, give an expression for small motions of the particle around the minima of Y(y).

Solution:
First we write out a power series in y to see behavior of y's acceleration and force.

$$\ddot{y} = -\frac{\partial V(x,y)}{\partial y} = \sin(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = y - \frac{y^3}{6} + ...$$

Note that sin(y) has translational symmetry, repeating every 2π. So this holds for every minimum, and we can replace y in the above equation with y'= y + 2nπ, with n an integer. This will allow us to cover all maxima and minima.

Part 1:
From the Taylor expansion above, we see that around y=0 (a maximum of Y(y)), the force on the particle has force proportional to the difference y. So the differential equation for the particle's motion is:

$$\frac{\partial^2 y}{\partial t^2} = y,\ y(0) = b,\ \dot{y}(0) = 0$$

We throw out higher order terms, because they are small for y's very close to a maximum. Then you see the general equation for y(t) is of the form

$$y(t) = A_+ e^t + A_- e^{-t}$$

Initial conditions give the values A+, A-, so we solve for y(t) around maxima as:

$$y(t) = \frac{b}{2}e^t + \frac{b}{2}e^{-t} = b\left(\frac{e^t + e^{-t}}{2}\right) = b\cosh(t)$$

Note that this does not work for long times, as the higher order terms become important as y increases.

Part 2:

For the second part of the problem, we solve the motion of the particle around minima of the potential. Truncating the power series for the force around these minima, we have the differential equation for y as:

$$\frac{\partial^2 y}{\partial t^2} = -y,\ y(0) = b,\ \dot{y}(0) = 0$$

The general solution for y(t) obeying this differential equation is:

$$y(t) = \mathrm{Re}\left[A_+ e^{it} + A_- e^{-it}\right] = A\cos(t+\phi)$$

The particular solution, with these initial conditions is:

$$y(t) = b\cos(t)$$

This is valid for small b's.

Alternative, easy method for both parts:

Model both as harmonic oscillator because force is proportional to y. Solving for the effective spring constant, we find:

$$k_{\text{eff.}} = \frac{\partial^2 Y(y)}{\partial y^2} = \begin{cases} 1 \text{ Minimum} \\ -1 \text{ Maximum} \end{cases}$$

Then we can find the effective angular frequency,

$$\omega = \sqrt{\frac{k}{m}} = \begin{cases} \pm i \text{ for maximum} \\ \pm 1 \text{ for minimum} \end{cases}$$

Then plugging this angular velocity in, we get the solution for both:

$$y(t) = b\cos(\omega t) = b\cos(\omega t) = \begin{cases} b\cosh(t) \text{ for maxima} \\ b\cos(t) \text{ for minima} \end{cases}$$

It should be noted that this solution is only valid for small b's, and in the case of the harmonic oscillator around the maxima, it is only valid for small times.