Damped oscillator with delta force

A particle of mass m is connected to a spring with spring constant k. Damping is added, proportional to the velocity. After being at rest for a long time, an impulse I is added to the particle. Find the position relative to the equilibrium position, x(t), for t > 0, given the system is:

a) critically damped

b) underdamped

c) overdamped

Solutions

Knowing general solution formulas for x(t), and our initial position and velocity conditions for various damping scenarios x(0) = 0 , v(0) = I/m , we can calculate formulas for velocity by taking the derivative of our general position formula, and we can use our initial conditions to solve for the coefficients A and B. Finally, plugging these back into our general solution gives us the final position function.

A delta force or transient force refers to when a force is applied instantaneously, and it implies that there is impulse given with no net change in position.

a) Critical Damping. When critically damped, $\omega_o = \frac{k}{m} =$ ß:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{I}{m}\partial(t)$$

$$x(t) = Ae^{-ßt} + Bte^{-ßt}$$

$$x(0) = 0 = A$$

$$\dot{x}(t) = Ae^{-ßt}\dot{(}-ß) + Bte^{-ßt}(-ß) + Be^{-ßt}$$

$$\dot{x}(0) = \frac{I}{m} = B$$

$$x(t) = \frac{I}{m}te^{-ßt}$$

b) Underdamping. When underdamped, $\omega_o = \frac{k}{m} > ß$:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{I}{m}\partial(t)$$

$$x(t) = Ae^{-ßt}\cos(\omega' t) + Be^{-ßt}\sin(\omega' t) \quad , \quad \omega' = \sqrt{\omega_o^2 - ß^2}$$

$$x(0) = 0 = A$$

$$\dot{x}(t) = Ae^{-ßt}\cos(\omega' t)(-ß) - Ae^{-ßt}\sin(\omega' t)\omega' + Be^{-ßt}\sin(\omega' t)(-ß) + Be^{-ßt}\cos(\omega' t)\omega'$$

$$\dot{x}(0) = \frac{I}{m\omega'} = B$$

$$x(t) = \frac{I}{m\omega'}e^{-ßt}\sin(\omega' t)$$

c) Overdamping. When overdamped, $\omega_o = \frac{k}{m} < ß$:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = \frac{I}{m}\partial(t)$$

$$x(t) = Ae^{-ß_+ t} + Be^{-ß_- t} \quad , \quad ß_o = ß \pm \sqrt{ß^2 - \omega_o^2}$$

$$x(0) = 0 = A + B$$

$$\dot{x}(t) = Ae^{-ß_+ t}(-ß_+) + Be^{-ß_- t}(-ß_-)$$

$$\dot{x}(0) = \frac{I}{m} = -ß_+ A - ß_- B$$

$$A = -B \rightarrow \frac{I}{m} = ß_+ B - ß_- B$$

$$B = \frac{I}{m(ß_+ - ß_-)}$$

$$A = -\frac{I}{m(ß_+ - ß_-)}$$

$$x(t) = -\frac{I}{m(ß_+ - ß_-)} e^{-ß_+ t} + \frac{I}{m(ß_+ - ß_-)} e^{-ß_- t}$$