Harmonic Oscillator with a Periodic Force

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A harmonic oscillator is driven by a periodic force given by

$$F(t) = \frac{3t}{\tau} \quad \text{for} \quad \frac{-\tau}{2} < t < \frac{\tau}{2}$$

where τ represents the period.

This force can be expressed as a Fourier series of the form

$$F(t) = \frac{f_0}{2} + \sum_{n>0} f_n \cos(n\omega t) + g_n \sin(n\omega t) \quad \text{for} \quad \omega \equiv \frac{2\pi}{\tau}$$

(a) Which coefficients will be zero?

We know that f_0 and f_n will be zero for all n because the function is odd.

(b) Express the non-zero coefficients as sums The general expression for g_n is

$$g_n = \frac{2}{\tau} \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} \mathrm{d}t F(t) \sin(n\omega t)$$

We now plug in our function $F(t)$

$$g_n = \frac{2}{\tau} \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} \mathrm{d}t \frac{3t}{\tau} \sin(n\omega t)$$

This integral is most conveniently done using integration by parts

Recall: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$

$$u \equiv t \qquad v \equiv \frac{-1}{n\omega} \cos(n\omega t)$$
$$\mathrm{d}u \equiv \mathrm{d}t \qquad \mathrm{d}v \equiv \mathrm{d}t \sin(n\omega t)$$

We then apply this to g_n and find that

$$g_n = \frac{6}{\tau^2} \left(\frac{-t}{n\omega} \cos n\omega t + \int \frac{\mathrm{d}t}{n\omega} \cos(n\omega t) \right) \Bigg|_{\frac{-\tau}{2}}^{\frac{\tau}{2}}$$

Simplify

$$g_n = \frac{6}{\tau^2}\left(\frac{-t}{n\omega}\cos n\omega t - \frac{1}{n^2\omega^2}\sin(n\omega t)\right)\Bigg|_{\frac{-\tau}{2}}^{\frac{\tau}{2}}$$

Evaluate

$$g_n = \frac{-3}{n\pi}\cos(n\pi)$$

Recall: $\omega \equiv \frac{2\pi}{\tau}$

(c) Express the force $F(t)$ as a sum

$$F(t) = \sum_{n>0} \frac{-3}{n\pi}\cos(n\pi)\sin(n\omega t)$$