Symmetry and Conservation Laws Practice Problems

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1. Consider a particle of mass m moving in a potential of the form

$$V(x, y, z) = V_0(x^2 + y^2)$$

What are the conserved quantities of the particle?

(a) p_x
(b) p_y
(c) p_z
(d) L_x
(e) L_y
(f) L_z
(g) E
(h) $|\vec{p}|$
(i) $|\vec{L}|$

Solution: there are three conserved quantities in this problem: p_z, L_z, and E.

- $\underline{p_z}$: the main reasoning behind why p_z is conserved is because $V(x, y, z)$ does not have an explicit dependence on z. This means that small changes in the position with respect to z will not alter the potential of the particle, so momentum in the z-direction is conserved.
- $\underline{L_z}$: the main reasoning behind why L_z is conserved is because $V(x, y, z)$ has rotational symmetry about the z-axis. This means that small rotations about the z-axis will not result in a difference in potential of the particle, meaning that the angular momentum of the particle in the z-direction is conserved.
- $\underline{E}$: the main reasoning behind why E is conserved is because $V(x, y, z)$ does not have an explicit dependence on t, so the total energy of the particle is not changing over time. Therefore, E is conserved.

- Other quantities: $V(x, y, z)$ depends on x and y, so p_x and p_y are not conserved. This also means that $|\vec{p}|$ is not conserved as a result. Additionally, $V(x, y, z)$ does not have rotational symmetry about the x axis or y axis, so L_x and L_y are not conserved. This also means that $|\vec{L}|$ is not conserved as well.

2. Consider a particle of mass m moving in a potential of the form

$$V(x, y, z) = \frac{-V_0}{\sqrt{r}}$$

Here, V_0 is just a constant. Which of the following quantities are conserved?

(a) p_x

(b) p_y

(c) p_z

(d) L_x

(e) L_y

(f) L_z

(g) E

(h) $|\vec{p}|$

(i) $|\vec{L}|$

Solution: there are five conserved quantities: E, L_x, L_y, L_z, and $|\vec{L}|$.

- E: the main reason why E is conserved is that there is no explicit dependence on time.
- L_x, L_y, L_z, and $|\vec{L}|$: the main reasoning why each of these components are conserved is that the potential is spherically symmetric. This means that you can rotate about the x, y, and z axes and the graph potential would not change. Since $|\vec{L}| = \sqrt{L_x^2 + L_y^2 + L_z^2}$, and each of the components of angular momentum are conserved, the magnitude of the angular momentum vector is also conserved.
- Other quantities: since r is the length vector, $r = \sqrt{x^2 + y^2 + z^2}$. $V(r)$ does depend on x, y, and z. This means that p_x, p_y, and p_z are not conserved and additionally, $|\vec{p}|$ is not conserved.

3. Consider a particle of mass m moving in a potential of the form

$$V(x, y, z) = \frac{V_0}{1 + \tan^2(yz) - \sec^2(yz) + \alpha^2}$$

Here, α is just a constant. What are the conserved quantities of the particle?

(a) p_x
(b) p_y
(c) p_z
(d) L_x
(e) L_y
(f) L_z
(g) E
(h) $|\vec{p}|$
(i) $|\vec{L}|$

Solution: the answer is that all of the quantities are conserved! Using trigonometry identities, we can reduce the denominator to

$$1 + \tan^2(yz) - \sec^2(yz) + \alpha^2 = \sec^2(yz) - \sec^2(yz) + \alpha^2 = \alpha^2$$

This means that

$$V(x, y, z) = \frac{V_0}{\alpha^2}$$

so the potential is constant. Therefore, all of the quantities are conserved.