Zitzelberger
Sundberg

Practice Exam Problem

Lagrangian With Normal Modes Problem

Consider 2 particles of mass m connected by 3 springs between two walls as shown in the figure below. Each spring has an unstretched length l and a spring constant k, and the total length of the system is $3l$.

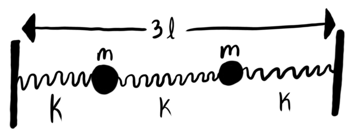

Figure 1: A picture of the system described above

1. Find the Lagrangian of the system in terms of the position of the masses, x_1 and x_2.
   Answer:
$$T = \frac{1}{2}m\dot{x_1}^2 + \frac{1}{2}m\dot{x_2}^2$$
$$U = \frac{1}{2}k(x_1 - l)^2 + \frac{1}{2}k(x_2 - x_1 - l)^2 + \frac{1}{2}k(x_2 - 2l)^2$$
$$L = T - U$$

2. Find the equations of motion. You may use any generalized coordinate system, as long as it accurately reflects the system.
   Answer:

   Define generalized coordinates:
   $q_1 = x_1 - l$
   $q_2 = x_2 - 2l$
$$L = \frac{1}{2}m\dot{q_1}^2 + \frac{1}{2}m\dot{q_2}^2 - \frac{1}{2}kq_1^2 - \frac{1}{2}k(q_2 - q_1)^2 - \frac{1}{2}kq_2^2$$

   Plugging this into the Euler-Lagrange equations give:
$$m\ddot{q}_1 = -kq_1 + k(q_2 - q_1)$$
$$m\ddot{q}_2 = -kq_2 - k(q_2 - q_1)$$

   Thus the equations of motion are:
$$\ddot{q}_1 = -\omega_0^2(2q_1 - q_2)$$
$$\ddot{q}_2 = -\omega_0^2(2q_2 - q_1)$$

3. Find the eigenfrequencies ω and normal modes (A/B or as an eigenvector) of the system.
   Answer:

   Start by guessing at the form of the answer:

$$q_1 = Ae^{i\omega t}, \quad q_2 = Be^{i\omega t}$$

   From these guesses, we then know:

$$\ddot{q_1} = -\omega^2 q_1, \quad \ddot{q_2} = -\omega^2 q_2$$

   Rewrite equations of motion so they can be converted to matrix-vector multiplication:

$$\ddot{q_1} = -2\omega_0^2 q_1 - \omega_0^2 q_2$$

$$\ddot{q_2} = \omega_0^2 q_1 - 2\omega_0^2 q_2$$

$$\begin{bmatrix} \ddot{q_1} \\ \ddot{q_2} \end{bmatrix} = \begin{bmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

   We know $\ddot{q_2}$ and $\ddot{q_2}$, so:

$$-\omega^2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

   The eigenfrequencies are the eigenvalues of the matrix. We now solve for the eigenvalues by inspection and get:

$$\omega_+ = \omega_0, \;\; \omega_- = \omega_0\sqrt{3}$$

   The normal modes of the system are the eigenvectors of the matrix, so we get:

$$v_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   This implies the general solutions of the system are:

$$q_+ = A_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_+ t}, q_- = A_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{i\omega_- t}$$

   Where A_+ and A_- are arbitrary amplitudes.

   Alternatively, the solutions can be found using traditional algebraic methods, though it's more tedious. Firstly, substitute our guess at the solutions into the equations to get the following:

$$-A\omega^2 = -2A\omega_0^2 + B\omega_0^2$$

$$-B\omega^2 = A\omega_0^2 - 2B\omega_0^2$$

For both equations, divide both sides by B, this yields:

$$-A/B\omega^2 = -2A/B\omega_0^2 + \omega_0^2$$

$$-\omega^2 = A/B\omega_0^2 - 2\omega_0^2$$

Continuing the algebra, the bottom equation can be easily manipulated into giving us an equation for A/B

$$A/B = -\omega^2/\omega_0^2 + 2$$

Substituting this into the first equation gives us this:

$$\omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4$$

Which can be solved quadratically for two solutions of ω^2, and therefore ω:

$$\omega_+ = \omega_0, \;\; \omega_- = \omega_0\sqrt{3}$$

Finally, Plugging these back into the A/B gives us two A/B, one where A=-B and A=B, These are, respectively:

$$A/B = 1, \quad A/B = -1$$

Note that these are the first component of the normal mode vectors, and each corresponds to a particular way that the system can move.