your name____________________________

Physics 321 Quiz #6 - Wednesday, Oct. 28

You can work in groups of 4 for this quiz. You should turn in one quiz for your group, with all four names. You may consult the internet, notes, or books to your heart's content.

Consider a particle of mass m in an attractive potential,

$$V(r) = -\frac{\alpha}{r}.$$

We had derived the following form for a trajectory,

$$r = \frac{1}{m\alpha/L^2 + A\cos(\theta - \theta_0)}.$$

For all the following problems consider the case where $\theta_0 = 0$, i.e. the minimum radius occurs at $\theta = 0$.

1. (2pts) Considering the case where $A > m\alpha/L^2$ (the scattering problem), express the total energy in terms of A, m, L and α.
2. (1 pt) If one rewrites the trajectory as

   $$r = \frac{r_0}{1 + \beta\cos\theta},$$

   express r_0 and β in terms of A, m, L and α.
3. (2pts) Express the trajectory as an equation in terms of x and y instead of r and θ. Show that the expression is a hyperbola, i.e., the form is

   $$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

   Express x_0, a and b in terms of r_0 and β.

Solutions:

1)

$$\begin{aligned}
r_{\rm min} &= \frac{1}{m\alpha/L^2 + A},\\
E &= -\frac{\alpha}{r_{\rm min}} + \frac{L^2}{2mr_{\rm min}^2},\\
&= -\alpha(m\alpha/L^2 + A) + \frac{L^2}{2m}(m\alpha/L^2 + A)^2\\
&= -\frac{m\alpha^2}{2L^2} + \frac{A^2L^2}{2m}.
\end{aligned}$$

2)

$$\begin{aligned}
r &= \frac{r_0}{1 + \beta\cos\theta},\\
r_0 &= \frac{L^2}{m\alpha},\ \ \beta = \frac{AL^2}{m\alpha} = Ar_0.
\end{aligned}$$

3)

$$
\begin{aligned}
1 &= \frac{r_0}{r+\beta x}, \\
r &= r_0 - \beta x \\
x^2 + y^2 &= r_0^2 - 2\beta x r_0 + \beta^2 x^2, \\
x^2(1-\beta^2) + 2\beta r_0 x + y^2 &= r_0^2, \\
(1-\beta^2)\left(x + \frac{\beta r_0}{1-\beta^2}\right)^2 + y^2 &= r_0^2 + \frac{\beta^2 r_0^2}{(1-\beta^2)}, \\
(1-\beta^2)\left(x + \frac{\beta r_0}{1-\beta^2}\right)^2 + y^2 &= \frac{r_0^2}{1-\beta^2}, \\
x_0 &= r_0 \frac{\beta}{\beta^2 - 1}, \\
\frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2} &= 1, \\
a^2 &= \frac{r_0^2}{(1-\beta^2)^2}, \\
b^2 &= \frac{r_0^2}{\beta^2 - 1}.
\end{aligned}
$$

Note that for a hyperbola $\beta > 1$, whereas for an ellipse, $\beta < 1$.