your name(s)____

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Physics 321 Quiz #7 - Friday, April 6 2018

Work in groups of 4 or less.
You wish to find the fastest route for a paddle boat, which moves with speed V_0 on still water, to travel a distance L, from $x = y = 0$ to $x = L, y = 0$. There is a current also moving in the x direction, but depending on y. The current's speed is $w_x = \alpha y$. The time for the voyage can be expressed as

$$t = \int_0^L \frac{dx}{dx/dt},$$

Our goal is use the Euler-Lagrange equation to solve for $y(x)$.

1. (10 pts) Express dx/dt in terms of w_x, V_0 and dy/dx.
2. (10 pts) Using the expression for t above, write a 2nd-order differential equation for $y(x)$ from the Euler-Lagrange equations. The equation should be in terms of L, α and w_x, y, d^2y/dx^2 and dy/dx.
3. (20 extra credit quiz points) Write a program to solve the Euler-Lagrange equation. Use $\alpha = 0.02$ s-1, $L = 1000$ m and $V_0 = 10$ m/s. What is the maximum value of y for this trajectory?

Extra-credit portion can be turned in Monday – no more than 4 names on a paper, which should include printout of program and the value of $y_{\rm max}$.

$$t = \int \frac{dx}{v_x} \quad , \quad \left(\frac{dx}{dt} - \cancel{w}\right)^2 + \left(\frac{dy}{dt}\right)^2 = V_0^2$$

$$t = \int_0^L \frac{dx}{\frac{dx}{dt}} =$$

$$\left(\frac{dx}{dt}\right)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right) - 2w\left(\frac{dx}{dt}\right) + w^2 - v_0^2 = 0$$

$$\frac{dx}{dt} = \frac{w \pm \sqrt{w^2 - (w^2 - v_0^2)(1+y'^2)}}{1 + (y')^2}$$

$$t = \int_0^L \frac{dx \, (1 + y'^2)}{\left(\alpha y \pm \sqrt{\alpha^2 y^2 - (\alpha^2 y^2 - v_0^2)(1+y'^2)}\right)}$$

$$f = \frac{(1 + y'^2)}{[\alpha y + R]} \quad , \quad R = \sqrt{-\alpha^2 y^2 y'^2 + V_0^2(1+y'^2)}$$

$$\partial_y f = \frac{-f}{[\alpha y + R]} \left\{ \alpha - \frac{1}{R} \alpha^2 y'^2 y \right\}$$

$$\partial_{y'} f = \frac{2y'}{(\alpha y + R)} - \frac{(f/R)}{(\alpha y + R)} \left[V_0^2 y' - \alpha^2 y^2 y' \right]$$

$$f = \frac{(1+y\ )}{[\alpha y + R]} \qquad R = \sqrt{V_0^2(1+y'^2) - \alpha^2 y^2 y}$$

$$\partial_y f = \frac{-f}{[\alpha y + R]}\left\{\alpha - \frac{1}{R}\alpha^3 y'^2 y\right\}$$

$$\partial_{y'} f = \frac{2y'}{(\alpha y + R)} - \frac{(f/R)}{(\alpha y + R)}\left[V_0^2 y' - \alpha^2 y^2 y'\right]$$

$$\partial_y f = \frac{-f}{(\alpha y + R)}\left\{\alpha + \frac{1}{2R}\partial_y R^2\right\}$$

$$\partial_{y'} f = \frac{2y'}{(\alpha y + R)} - \frac{(f/2R)}{(\alpha y + R)}\partial_{y'} R^2$$

$$\partial_y R^2 = -2\alpha^2 y'^2 y$$

$$\partial_{y'} R^2 = 2V_0^2 y' - 2\alpha^2 y^2 y'$$

$$\frac{d}{dx}(\partial_{y'} f) = \frac{2y''}{(\alpha y + R)} - \frac{2y'}{(\alpha y + R)^2}\left[\alpha y' + \frac{1}{2R}y'(\partial_y R^2) + \frac{1}{2R}y''(\partial_{y'} R^2)\right]$$

$$- \frac{\frac{1}{2R}(\partial_{y'} R^2)}{\alpha y + R}\left[(\partial_y f)y' + (\partial_{y'} f)y''\right]$$

$$+ \frac{f}{4R^3}\frac{(\partial_{y'} R^2)}{(\alpha y + R)}\left[(\partial_y R^2)y' + (\partial_{y'} R^2)y''\right]$$

$$- \left(\frac{f/2R}{\alpha y + R}\right)\left(2V_0^2 y'' - 4\alpha^2 y\, y'^2 - 2\alpha^2 y^2 y''\right)$$

$$+ \frac{(f/2R)(\partial_{y'} R^2)}{(\alpha y + R)^2}\left[\alpha y' + (\partial_y R)y' + (\partial_{y'} R)y''\right]$$

$$\partial_y f = \frac{d}{dx}\partial_{y'} f$$

Another approach:

$$\text{MAX} \int dt\; v_x(t)$$

$$v_x = \alpha y + \sqrt{V_0^2 - y'^2}$$

$$\frac{\partial v_x}{\partial y} = \alpha \;;\quad \frac{\partial v_x}{\partial y'} = \frac{-y'}{\sqrt{V_0^2 - y'^2}}$$

$$\alpha = \frac{-y''}{\sqrt{V_0^2 - y'^2}} - \frac{y'^2}{(V_0^2 - y'^2)^{3/2}}\, y''$$

$$\boxed{\alpha = \frac{-V_0^2 \quad y''}{(V_0^2 - y'^2)^{3/2}}}$$

$$\frac{\alpha}{V_0^2}\, t = -\int \frac{dv_y}{(V_0^2 - v_y^2)^{3/2}}$$

$$\alpha t = -\int_{(v_y/V_0)_0}^{(v_y/V_0)} \frac{du}{(1-u^2)^{3/2}} = \left.\frac{-u}{\sqrt{1-u^2}}\right|_{v_{y0}/V_0}^{v_y/V_0}$$

$$u \equiv v_y/V_0 \,,\quad u_0 \equiv v_y(t=0)/V_0$$

$$\frac{u}{\sqrt{1-u^2}} = -\alpha t + \frac{u_0}{\sqrt{1-u_0^2}}$$

$$u = \frac{u_0/\sqrt{1-u_0^2} - \alpha t}{\sqrt{1 + \left(\frac{u_0}{\sqrt{1-u_0^2}} - \alpha t\right)^2}} = \frac{v_y}{V_0}$$

$$v_y = \frac{u_0/\sqrt{1-u_0^2} - \alpha t}{\sqrt{1 + \left(\frac{u_0}{\sqrt{1-u_0^2}} - \alpha t\right)^2}} V_0 \quad , \quad u_0 = \frac{v_y(t=0)}{V_0}$$

$$y = -\sqrt{1 + \left(\frac{u_0}{\sqrt{1-u_0^2}} - \alpha t\right)^2} \cdot \frac{V_0}{\alpha} + \sqrt{\frac{1}{1-u_0^2}} \frac{V_0}{\alpha}$$

$$v_x = \alpha y + \sqrt{V_0^2 - v_y^2}$$

$$= \alpha y + V_0 \left\{ 1 - \frac{\left(\frac{u_0}{\sqrt{1-u_0^2}} - \alpha t\right)^2}{1 + \left(\frac{u}{\sqrt{1-u_0^2}} - \alpha t\right)^2} \right\}^{1/2}$$

$$= V_0 \frac{1}{\sqrt{1 + \left(\frac{u_0}{\sqrt{1-u_0^2}} - \alpha t\right)^2}} + \alpha y$$

$$\gamma \equiv u_0/\sqrt{1-u_0^2} \quad , \quad u_0 \equiv v_y(t=0)/V_0$$

$$v_y = \frac{(\gamma - \alpha t) V_0}{(1 + (\gamma - \alpha t)^2)^{1/2}}$$

$$y = \sqrt{1+\gamma^2} \frac{V_0}{\alpha} - \sqrt{1 + (\gamma - \alpha t)^2} \frac{V_0}{\alpha}$$

$$v_x = \alpha y + V_0 \frac{1}{(1 + (\gamma - \alpha t)^2)^{1/2}}$$

$$v_x = V_0 \sqrt{1+\gamma^2} - V_0 \frac{(\gamma - \alpha t)^2}{\sqrt{1 + (\gamma - \alpha t)^2}}$$

$$v_x = V_0\sqrt{1+\gamma^2} - V_0 \frac{(\gamma-\alpha t)^2}{\sqrt{1+(\gamma-\alpha t)^2}}$$

$$\int \frac{u^2\,du}{\sqrt{1+u^2}} \overset{?}{=} \int \sinh^2\theta \, d\theta \qquad u = \sinh\theta$$

$$= \frac{1}{2}\int (\cosh 2\theta - 1)\, d\theta$$

$$= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta$$

$$= -\frac{1}{2}\sinh^{-1} u + \frac{1}{2} u\sqrt{1+u^2}$$

$$x = V_0\sqrt{1+\gamma^2}\, t - V_0 \frac{\sinh^{-1}(\gamma-\alpha t)}{2\alpha} + \frac{v_0}{2\alpha}(\gamma-\alpha t)\sqrt{1+(\gamma-\alpha t)^2} + \frac{v_0}{2\alpha}\sinh^{-1}\gamma - \frac{v_0}{2\alpha}\gamma\sqrt{1+\gamma^2}$$

Now one must solve for γ. this is done by trial and error until one find γ s.t. there exists a time t_f, where

$$x(\gamma, t_f) = 1000 \quad \& \quad y(\gamma, t_f) = 0$$

By inspecting $y(t)$, one can see that if it is to be symmetric about $t_f/2$, then $t_f = 2\gamma/\alpha$.

$\therefore$ simply adjust γ until

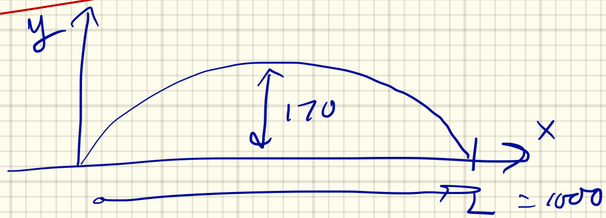
$$x(\gamma, t = 2\gamma/\alpha) = L.$$

For $L = 1000$, $\alpha = 0.02$, $V_0 = 10$,

$$\gamma = 0.89267$$

$$t_f = 89.267 \text{ s}$$

$$y(t_f/2) = 170.24$$



save 10.8 s by not going straight.