your name(s)____

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Physics 321 Quiz #11 - Wednesday/Friday, April 3/5 2019

Work in groups of 3 or less.
You wish to find the fastest route for a paddle boat, which moves with speed V_0 on still water, to travel a distance L, from $x = y = 0$ to $x = L, y = 0$. There is a current also moving in the x direction, but depending on y. The current's speed is $w_x = \alpha y$. Rather than minimizing the time to go a distance L, we will maximize the distance L one travels in a time T, while returning to $y = 0$ at time T. This will result in the same trajectory once the time is adjusted so the distance is indeed L. That distance is

$$L = \int_0^T dt \frac{dx}{dt}(y, y'), \quad y' \equiv dy/dt$$

Our goal is use Euler's equation to solve for $y(t)$.

1. (5 pts) Express $x' = dx/dt$ in terms of α, V_0, y and y'.
2. (5 pts) Using the expression for x above, write a 2nd-order differential equation for $y(t)$ using Euler's equation. The equation should be in terms of α, V_0, y, d^2y/dt^2 and dy/dt.
3. (5 pts) Show that the solution has the form

$$y = y_0 - \frac{V_0}{\alpha}\sqrt{1 + \alpha^2(t - t_0)^2},$$

   where y_0 and t_0 are the two arbitrary constants.
4. (5 pts) For an optimal path for a time total time T, in terms of α, V_0 and T.
   (a) What is t_0? (Use symmetry arguments)
   (b) What is y_0?

FYI: If you wanted to find the length for such a trip, the result would be:

$$L = \frac{V_0 T}{2}\sqrt{1 + (\alpha T/2)^2} + \frac{V_0}{\alpha}\sinh^{-1}(\alpha T/2).$$

Finding T in terms of L involves solving a transcendental equation. If you want to find $x(t)$, you can integrate $x'(t)$. It's messy, but you would get:

$$\begin{aligned} x(t) &= \frac{V_0 T}{4}\left[R(t) - R(0)\right] + \frac{V_0}{2\alpha}\left[\sinh^{-1}(\alpha(t - T/2)) + \sinh^{-1}(\alpha T/2)\right] + V_0 t\left[R(0) - R(t)/2\right], \\ R(t) &\equiv \sqrt{1 + \alpha^2(t - T/2)^2}. \end{aligned}$$