

7 Landau Field Theory

Nothing is real and nothing to get hung about, Strawberry Fields forever - J. Lennon, P. McCartney

7.1 What are fields?

Field theory represents a means by which the state of a system is expressed in terms of a quantity ϕ which is a function of the position \vec{x} . The quantity ϕ represents a measure of some quantity averaged over a small volume δV . One example of such a quantity is the average magnetization per unity volume, $\vec{m}(\vec{x})$, while another is the number density $\rho(\vec{x})$. In each case, when viewed microscopically, the magnetization density and the number density are not smooth quantities, but are carried by point-like objects. Thus, field theories are only applicable when looking at averages or correlations on scales larger than the microscopic scale of the matter. As will be seen further ahead, correlation length scales tend to go to infinity near a critical point, a region where field theories are especially valuable.

Finite-temperature field theories are usually expressed in terms of the free-energy density, $f(\vec{x})$,

$$\begin{aligned} F &\equiv \int d^3x f(\phi, \nabla\phi), \\ f &= \frac{\kappa}{2}(\nabla\phi)^2 + \mathcal{V}(\phi). \end{aligned} \tag{1}$$

The second statement, that derivatives appear only quadratically, is certainly an ansatz, but as we will see below, the assumption is motivated from physical arguments. The choice of F for the free energy should not suggest that one is always interested in the Helmholtz free energy, as for some applications it might represent the Gibb's free energy or the entropy. If the parameter ϕ changes slowly compared to the time required to come to thermal equilibrium (constant temperature), the use of the Helmholtz free energy is justified, and if the pressure also equilibrates quickly compared to ϕ , the Gibb's free energy would be the appropriate choice.

The gradient term is warranted whenever there is an attractive short-range interaction. As an example, one can look at the two-dimensional $X - Y$ model, where spins on a lattice are denoted by an angle θ , with the following interaction between nearest neighbors,

$$V_{ij} = V_0[1 - \cos(\theta_i - \theta_j)]. \tag{2}$$

For small differences in the angle this can be approximated as:

$$\begin{aligned} V &= \frac{V_0}{2} \sum_{i_x, i_y} (\theta_{i_x} - \theta_{i_x-1})^2 + (\theta_{i_y} - \theta_{i_y-1})^2 \\ &= \frac{V_0}{2} \sum_{i_x, i_y} [(\theta_{i_x} - \theta_{i_x-1})^2 + (\theta_{i_y} - \theta_{i_y-1})^2] \frac{(\Delta L)^2}{(\Delta L)^2} \\ &= \frac{V_0}{2} \int dx dy (\nabla\theta)^2, \end{aligned} \tag{3}$$

where ΔL is the lattice spacing. Thus, in this example, it is easy to identify κ with the microscopic quantity V_0 . Since the free energy is $F = E - TS$, terms that contribute to the energy density carry through to the free energy density.

A second example is that of a liquid, or gas, where ϕ refers to the density $\rho(\vec{x})$. In that case the attractive interaction is mainly the two-particle potential $v(\vec{r})$,

$$V = \int d^3x d^3x' \langle \rho(\vec{x})\rho(\vec{x}') \rangle v(\vec{x} - \vec{x}'). \tag{4}$$

For a field theory, we are interested in a average density at a given point $\bar{\rho}(\vec{x})$. In terms of the averages, the two-body potential energy is

$$V = \int d^3x d^3x' \bar{\rho}(\vec{x}) \bar{\rho}(\vec{x}') C(\vec{x}, \vec{x}') v(\vec{x} - \vec{x}'), \quad (5)$$

$$C(\vec{x}, \vec{x}') \equiv \frac{\langle \rho(\vec{x}) \rho(\vec{x}') \rangle}{\bar{\rho}(\vec{x}) \bar{\rho}(\vec{x}')}.$$

Here, $C(\vec{r})$ is the correlation (unity if uncorrelated) for two particles to be separate by \vec{r} . In the low density limit (particles interacting only two-at-a-time) the correlation is approximately

$$C(\vec{x}, \vec{x}') \approx e^{-v(\vec{r})/T}. \quad (6)$$

Next, one can make a gradient expansion for $\bar{\rho}$,

$$\bar{\rho}(\vec{x}') \approx \bar{\rho}(\vec{x}) + (x'_i - x_i) \partial_i \bar{\rho}(\vec{x}) + \frac{1}{2} \sum_{i,j} (x'_i - x_i)(x'_j - x_j) \partial_i \partial_j \bar{\rho}(\vec{x}). \quad (7)$$

Since $V(\vec{r})$ is an even function of \vec{r} , the odd terms in the expansion can be discarded leaving

$$V = \int d^3x \left\{ \mathcal{V}_{\text{local}}(\bar{\rho}(\vec{x})) - \frac{\kappa}{2} \bar{\rho}(\vec{x}) \nabla^2 \bar{\rho}(\vec{x}) \right\} \quad (8)$$

$$\kappa = -\frac{1}{3} \int d^3r r^2 e^{-v(r)/T} v(r), \quad (9)$$

where the approximation for the correlation, Eq. (6), has been used for the correlation function. The factor of 1/3 in the expression for κ comes from assuming that $v(\vec{r})$ is rotationally symmetric, so that the integral with r_x^2 is one third the integral with r^2 . The term $\mathcal{V}_{\text{local}}$ is the contribution to the potential energy when density gradients are neglected. Assuming that other contributions to the energy do not contribute to the ∇^2 term, the free energy becomes:

$$F = \int d^3x \left[\mathcal{V}(\bar{\rho}(\vec{x})) - \frac{\kappa}{2} \bar{\rho}(\vec{x}) \nabla^2 \bar{\rho}(\vec{x}) \right] \quad (10)$$

$$= \int d^3x \left[\mathcal{V}(\bar{\rho}(\vec{x})) + \frac{\kappa}{2} (\nabla \bar{\rho})^2 \right]. \quad (11)$$

where the last step involved integrating by parts. Since the definition of density usually implies an average the “bar” over $\bar{\rho}$ can be dropped .

An obvious question concerns how one might justify dropping higher order gradients in the Taylor expansion of $\rho(\vec{x}')$. Certainly, such terms should exist. Calculating such higher-order terms would involve finding expressions for coefficients like were found for κ in Eq. (9) above. However, rather than one extra power of r^2 as in Eq. (9), one would find powers of $r^4, r^6 \dots$. If the potential $v(r)$ falls off exponentially with a sufficiently short range, these terms can be neglected, and even if they fall off with a power-law (such as $1/r^6$ for the Lennard-Jones potential) screening effects should give an exponential fall-off and result in an exponential behavior at large r to keep the coefficients finite.

One common mistake students make when seeing a mean-field theory is to assume that the gradient term originated from kinetic terms in the microscopic equations of motion. Instead, such terms originate from attractive short-range interactions. In fact, if the interactions were repulsive κ would have the wrong sign. The terms do, however, represent kinetic-energy type terms in the free energy. For instance, if V is quadratic, one can express the free energy as a sum of independent contributions from momentum modes where $\delta\rho \sim \sin(kx)$. In that case, the ∇^2 term in the free energy behaves as k^2 , just as it would for free particles or sound modes.

7.2 Calculating surface energies in field theory

As discussed in Chapter 3, two phases can co-exist in equilibrium. However, there is a penalty associated with the boundary. This is often referred to as a surface energy or a surface free energy. To calculate the surface free energy, we consider a large box where the left side ($x < 0$) is in a gaseous state and the right side ($x > 0$) is in a liquid state. If A is the area of the $y - z$ plane, the total Helmholtz free energy ($F = E - TS$) is

$$F/A = \int dx \left[\mathcal{V}(\rho(x)) + \frac{\kappa}{2}(\partial_x \rho)^2 \right]. \quad (12)$$

Our goal is to find the density profile that minimizes the free energy given the constraint that it has the density of the gas at $x = -\infty$ and the density of the liquid at $x = \infty$.

The Helmholtz free energy is the quantity one should minimize if a system is connected to a heat bath and the number of particles in the box is fixed. If the system can also exchange particles with a bath with chemical potential μ_0 , one should include the free energy associated with the particles that have been added or subtracted into the bath. In that case, the number is no longer conserved, and the quantity to be minimized is

$$F - \mu_0 N = A \int dx \left[\mathcal{V}(\rho(x)) - \mu_0 \rho + \frac{\kappa}{2}(\partial_x \rho)^2 \right] \quad (13)$$

$$= A \int dx \left[-P(\rho(x)) + (\mu - \mu_0)\rho + \frac{\kappa}{2}(\partial_x \rho)^2 \right]. \quad (14)$$

The last step used the thermodynamic identity that the free energy density for a uniform gas is $\mu\rho - P$. Figure 1 illustrates a plot of $-P(\rho, T) + (\mu - \mu_0)\rho$ as a function of the density for fixed temperature. By picking the appropriate value of μ_0 for the specified temperature T , the two minima are at the same level and phase coexistence can occur. In fact, at the minima $\mathcal{V} - \mu_0\rho = -P$, and the statement that both minima are equal is equivalent to saying that for the specified T , μ_0 was chosen so that two densities have the same pressure.

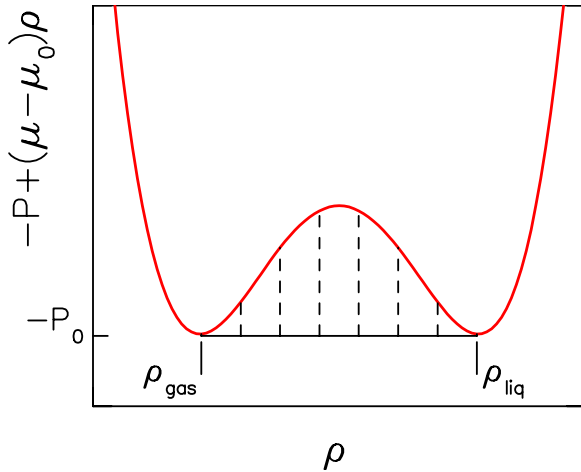


Figure 1: Here, $P(\rho, T)$ and $\mu(\rho, T)$ are the pressure and chemical potential of a gas at uniform density ρ and temperature T . If the system is connected to a heat bath with chemical potential μ_0 , the system will choose a density to minimize $-P + (\mu - \mu_0)\rho$. If μ_0 equals the chemical potential for phase coexistence, the two minima are degenerate and are located at the coexistence densities, ρ_{gas} and ρ_{liq} for the given T . A boundary between the two regions must traverse the region of less favorable free-energies, hence there is a surface free energy as described in Eq. (19). In this expression, the surface energy is found by integrating over the densities between the two coexistence values with the integrand being proportional to the square root of the height, illustrated by the dashed lines.

The cost of the net free energy for a profile $\rho(x)$ that differs from the two coexistence densities is,

$$\Delta F/A = \int dx \left[P_0 - P + (\mu - \mu_0)\rho + \frac{\kappa}{2}(\partial \rho)^2 \right], \quad (15)$$

where P_0 is the coexistence pressure for the temperature T . The quantity $(\mathcal{V} - \mu_0\rho + P_0)$ is represented by the height of the lines in the shaded region of Fig. 1. After replacing $dx(\partial_x\rho)^2 = d\rho(\partial_x\rho)$, the integration can be restated in terms of the densities,

$$\Delta F/A = \int_{\rho_{\text{gas}}}^{\rho_{\text{liq}}} d\rho \left[\frac{P_0 - P + (\mu - \mu_0)\rho}{\partial_x\rho} + \frac{\kappa}{2}\partial_x\rho \right]. \quad (16)$$

Next, one chooses $\partial_x\rho$ to minimize the free energy. Whereas the first term would be minimized by choosing a step function, $\partial_x\rho = \infty$, the second term would be minimized by a smooth profile. Minimizing functional integral yields,

$$\frac{d}{d(\partial_x\rho)} \left[\frac{P_0 - P + (\mu - \mu_0)\rho}{\partial_x\rho} + \frac{\kappa}{2}\partial_x\rho \right] = 0, \quad (17)$$

$$\partial_x\rho = \sqrt{2\frac{P_0 - P + (\mu - \mu_0)\rho}{\kappa}}, \quad (18)$$

$$\Delta F/A = \sqrt{2\kappa} \int_{\rho_{\text{gas}}}^{\rho_{\text{liq}}} d\rho \sqrt{P_0 - P + (\mu - \mu_0)\rho}. \quad (19)$$

Thus, the coefficient κ is responsible for the surface free energy. If there were no gradient term in the functional for the free energy, there would be no penalty and the density profile between equilibrated phases would become a step function. Looking back at the expression for κ in the previous section shows that surface energies will be larger for interactions with longer ranges.

EXAMPLE:

Let the function $\Delta\Psi \equiv P_0 - P + (\mu - \mu_0)\rho$ be approximated with the following form,

$$\Delta\Psi = \frac{B}{2} [(\rho - \rho_c)^2 - \alpha^2]^2.$$

Find the surface energy in terms of B , α and κ .

From Eq. (19), the surface energy is

$$\begin{aligned} F/A &= \sqrt{\kappa B} \int_{\rho - \rho_c = -\alpha}^{\rho - \rho_c = \alpha} d\rho \sqrt{[(\rho - \rho_c)^2 - \alpha^2]^2} \\ &= \sqrt{\kappa B} \int_{-\alpha}^{\alpha} dx (\alpha^2 - x^2) \\ &= \frac{2}{3}\alpha^3\sqrt{\kappa B} \end{aligned}$$

Note that B , κ and α^2 would depend on the temperature. Since two independent minima no longer exist for $T > T_c$, the coefficient α^2 goes to zero at T_c , which means that the surface energy also becomes zero.

7.3 Correlations and the critical point

The role of correlations, e.g.,

$$C(\vec{r} - \vec{r}') = \langle \phi^*(\vec{r})\phi(\vec{r}') \rangle, \quad (20)$$

is central to the study of critical phenomena. Note that here we have assumed a complex field, which has two independent components ϕ_r and ϕ_i . For this case,

$$\langle \phi^*(\vec{r})\phi(\vec{r}') \rangle = \langle \phi_r(\vec{r})\phi_r(\vec{r}') \rangle + \langle \phi_i(\vec{r})\phi_i(\vec{r}') \rangle. \quad (21)$$

Thus, if one were to solve for correlations of a one-component (real) field, one would divide the expression derived here for complex fields by a factor of two. We work with complex fields only because the Fourier transforms are somewhat less painful.

Near the critical point the characteristic length of such correlations approaches infinity. To demonstrate this, we consider a purely quadratic functional, as the correlations can be found analytically. The effect of higher-order terms alters the behavior quantitatively (e.g., different values for the critical exponents), but does not alter the qualitative behavior shown here. For quadratic potentials,

$$\mathcal{V}(\phi) = \frac{A}{2}\phi^2, \quad (22)$$

the problem can be divided into individual modes:

$$F = \int d^3r f = \int d^3r \left[\frac{A}{2}|\phi|^2 + \frac{\kappa}{2}|\nabla\phi|^2 \right] \quad (23)$$

$$= \frac{1}{2} \sum_{\vec{k}} (A + \kappa k^2) |\tilde{\phi}_{\vec{k}}|^2, \quad (24)$$

where the Fourier and inverse-Fourier transforms are defined by

$$\begin{aligned} \tilde{\phi}_{\vec{k}} &\equiv \frac{1}{\sqrt{V}} \int d^3r e^{i\vec{k}\cdot\vec{r}} \phi(\vec{r}), \\ \phi(\vec{r}) &= \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} \tilde{\phi}_{\vec{k}}. \end{aligned} \quad (25)$$

Since the modes are independent, and therefore uncorrelated, the correlation in terms of momentum components is straight-forward,

$$\begin{aligned} \langle \tilde{\phi}_{\vec{k}}^* \tilde{\phi}_{\vec{k}'} \rangle &= \delta_{\vec{k},\vec{k}'} \langle |\tilde{\phi}_{\vec{k}}|^2 \rangle \\ &= \delta_{\vec{k},\vec{k}'} \frac{T}{(A + \kappa k^2)}, \end{aligned} \quad (26)$$

where the equipartition theory was used for the last step.

One can now calculate the correlation in coordinate space,

$$\begin{aligned} \langle \phi^*(\vec{r}) \phi(\vec{r}') \rangle &= \frac{1}{V} \sum_{\vec{k},\vec{k}'} e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}'} \langle \tilde{\phi}_{\vec{k}}^* \tilde{\phi}_{\vec{k}'} \rangle \\ &= \frac{1}{V} \sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \frac{T}{(A + \kappa k^2)}. \end{aligned} \quad (27)$$

Now, the sum over modes can be transformed to an integral,

$$\langle \phi^*(\vec{r}) \phi(\vec{r}') \rangle = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \frac{T}{(A + \kappa k^2)}. \quad (28)$$

This integral can be performed by first integrating over the angle between \vec{k} and $(\vec{r} - \vec{r}')$, then performing a contour integration,

$$\begin{aligned} \langle \phi^*(\vec{r}) \phi(\vec{r}' = 0) \rangle &= \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} k dk \sin(kr) \frac{T}{(A + \kappa k^2)} \\ &= \frac{T}{4\pi\kappa r} e^{-r/\xi}, \quad \xi = \sqrt{\kappa/A}. \end{aligned} \quad (29)$$

Here, ξ is referred to as the correlation length. Near the critical point, the curvature of the free-energy w.r.t. the order parameter ϕ switches sign, and as A goes to zero, $\xi \rightarrow \infty$. The fact that this length becomes much larger than any driving microscopic length legitimizes the scaling arguments used in the discussion of critical phenomena in the next section.

In the case where ϕ refers to the magnetization density $m(\vec{r})$, the correlation function also provides the susceptibility,

$$\begin{aligned}
\chi &\equiv \frac{d}{d\mu B} \langle m(\vec{r}=0) \rangle & (30) \\
&= \frac{d}{d\mu B} \frac{\text{Tr} e^{-\beta H_0 + \int d^3 r \mu B m(\vec{r})} m(\vec{r}=0)}{\text{Tr} e^{-\beta H_0 + \int d^3 r \mu B m(\vec{r})}} \\
&= \frac{1}{T} \frac{\int d^3 r \text{Tr} e^{-\beta H_0} m(\vec{r}) m(\vec{r}=0)}{\text{Tr} e^{-\beta H_0}} - \frac{1}{T} \frac{[\text{Tr} e^{-\beta H_0} m(\vec{r}=0)] [\int d^3 r \text{Tr} e^{-\beta H_0} m(\vec{r})]}{[\text{Tr} e^{-\beta H_0}]^2} \\
&= \frac{1}{T} \int d^3 r \{ \langle m(\vec{r}) m(\vec{r}=0) \rangle - \langle m(\vec{r}=0) \rangle^2 \} & (31) \\
&= \frac{2}{\kappa} \xi^2,
\end{aligned}$$

where Eq. (29) was inserted into the expression for the correlation in the last step. Relations such as this are common in linear response theory, and often come under the catch-all phrase of the fluctuation-dissipation theorem. Linear response theory is that the change of an observable $\langle \delta A \rangle$ due to a small field that couples to a second observable B , $H_{\text{int}} = B \delta F$. This dependence can be expressed in terms of a generalized susceptibility, χ , where $\delta A = \chi \delta F$ defines χ . The central result of linear response theory is that χ can be expressed in terms of correlations of the type $\langle A(0) B(r) \rangle$. Although we will not have the time to cover linear response theory, I recommend reading the first two chapters of Forster¹. Equation (31) will be important in the next chapter as it will be used to connect the divergent behavior of the correlation function at the critical point to other quantities such as the susceptibilities and specific heat.

The curvature used in the free energy A changes sign at the critical point. One can parameterize this as:

$$A(T) = at, \quad t \equiv (T - T_c)/T_c. \quad (32)$$

Thus, the correlation length diverges as $t^{-1/2}$,

$$\begin{aligned}
\xi &= \sqrt{\frac{\kappa}{A}} = \sqrt{\kappa/a} t^{-1/2}, & (33) \\
\chi &= \frac{2T}{a} t^{-1}
\end{aligned}$$

The exponents describing the divergences here are known as critical exponents. In Landau theory, all three-dimensional systems would be have the same *critical exponents*,

$$\begin{aligned}
\xi &\sim |t|^{-\nu}, \quad \nu = 1/2, & (34) \\
\chi &\sim |t|^{-\gamma}, \quad \gamma = 1.
\end{aligned}$$

The specific heat is also divergent, but to see that we must first calculate the free energy. Equation (23) gives the free energy for a particular set of amplitudes $\phi_1, \phi_2 \dots$. To calculate the free energy of the entire system, one must sum over all the possible values of ϕ_i ,

$$\mathcal{F}_{\text{tot}} = -T \ln \sum_{\text{all configurations } \phi_1, \phi_2 \dots} e^{-F(\phi_1, \phi_2 \dots) / T}. \quad (35)$$

¹D. Forster, Hydrodynamic Fluctuations, Broken Symmetry, and Correlations Functions

Since F can be written as a sum over individual modes \vec{k} ,

$$\mathcal{F}_{\text{tot}} = -T \sum_{\vec{k}} \ln \left(\sum_{\phi_k} e^{-F_k(\phi_k)/T} \right). \quad (36)$$

As was shown in Chapter 3, the sum over states ϕ_k can be written as an integral over the real and imaginary parts of ϕ_k , so that the partition function becomes

$$\begin{aligned} e^{-F_k/T} &= \frac{1}{2\pi} \int d\phi_r d\phi_i e^{-(1/2)(A+\kappa k^2)|\phi_k|^2}, \\ &\sim (A + \kappa k^2)^{-1}, \end{aligned} \quad (37)$$

where the last expression ignores additive constants. Finally, the total free energy of the system will be

$$\mathcal{F}_{\text{tot}} = T \sum_{\vec{k}} \ln(A + \kappa k^2). \quad (38)$$

After inserting $A = at$, one can solve for the contribution to the specific heat from the temperature dependence in A . One must first take a derivative w.r.t. t to get an expression for the energy density, then take a second derivative to find the specific heat. The results is

$$C_V/V = \frac{a^2}{(2\pi)^3} \int d^3k \frac{1}{(at + \kappa k^2)^2} \sim t^{-1/2}. \quad (39)$$

This last term represents another critical exponent,

$$C_V \sim t^{-\alpha}, \quad \alpha = 1/2. \quad (40)$$

Another critical component concerns the rise of the mean magnetization for temperatures just below T_c . In our approximation that we consider only small fluctuations about the free energy minimum, our choice of the mean magnetization comes from the mean-field approximation from the previous chapter,

$$\langle m \rangle \sim |t|^\beta, \quad \beta = 1/2. \quad (41)$$

Finally, for an external field and $T = T_c$, mean field theory gives

$$\langle m \rangle \sim (\mu B)^{1/\delta}, \quad \delta = 3. \quad (42)$$

The divergence of the specific heat is related to the divergence of the integrand above when t is set to zero. Note that for d dimensions, the integrand goes as $dk k^{(d-1)}/k^4$, which will be divergent for $d \leq 3$. Thus, in four dimensions or higher, the specific heat is well defined at $t \rightarrow 0$, whereas the behavior is more complicated for the more physical two- and three-dimensional cases.

Summarizing the critical exponents in Landau theory,

$$\langle \phi(0)\phi(r) \rangle \sim r^{2-d} e^{-r/\xi}, \quad \xi \sim t^{-\nu}, \quad \nu = 1/2, \quad (43)$$

$$C_v \sim t^{-\alpha}, \quad \alpha = 2 - d/2, \quad (44)$$

$$\langle m \rangle (t = 0) \sim H^{1/\delta}, \quad \delta = 3, \quad (45)$$

$$\langle m \rangle (H = 0, t < 0) \sim t^\beta, \quad \beta = 1/2, \quad (46)$$

$$\chi \sim t^{-\gamma}, \quad \gamma = 1, \quad (47)$$

where d is the dimensionality. We will see in the next two sections that Landau theory is not valid near T_c for $d \leq 3$. Nonetheless, the features of Landau theory are qualitatively correct, and most important, the correct way of visualizing the behavior requires Landau theory for perspective.

7.4 Validity of Landau Theory Near T_c : The Ginzburg Criteria

The exponents above were based on an assumption that one could ignore the quartic piece of the free energy density, which would only be justified if fluctuations of the order parameter were small compared to the mean value. To see whether that is the case, we consider the quartic case

$$f(x) = \frac{A}{2}|\phi|^2 + \frac{B}{4}|\phi|^4, \quad (48)$$

where $A = at$ is negative. The free energy is then minimized for

$$\langle |\phi| \rangle^2 = A/B = at/B, \quad (49)$$

whereas the fluctuations behaved as

$$\langle \phi(0)\phi(r) \rangle \sim \frac{T}{\kappa} r^{2-d} e^{-r/\xi}. \quad (50)$$

The last expression was derived exactly for three dimensions previously, see Eq. (29). The integrals for the three-dimensional integration can be generalized for an arbitrary number of dimensions d in the limit that r is large, resulting in the expression above. If one uses $r = \xi$ as a characteristic length scale for comparison, the typical size for fluctuations are:

$$\langle \delta\phi\delta\phi \rangle_{\text{typical}} \sim \frac{T}{\kappa} \xi^{2-d}, \quad (51)$$

where dimensionless constants such as $(2\pi)^d$ are ignored. For mean-field theory to be valid, one must justify ignoring the quartic piece in the free energy above, or

$$\begin{aligned} \frac{A}{2} \langle \delta\phi\delta\phi \rangle_{\text{typical}} &>> \frac{B}{4} \langle \delta\phi\delta\phi \rangle_{\text{typical}}^2, \\ at &>> B \frac{T}{\kappa} \xi^{2-d}. \end{aligned} \quad (52)$$

Since

$$\xi \sim \sqrt{\kappa/at}^{-1/2}, \quad (53)$$

the criteria for validity becomes

$$t^{2-d/2} >> BT_c \kappa^{-d/2} a^{-2+d/2}. \quad (54)$$

As the critical point is approached $t \rightarrow 0$, and the criteria will be satisfied for $d > 4$. However, mean field theory is unjustified in the critical region for three dimensions or fewer.

Despite the fact that the theory is unjustified, Landau theory is still enormously useful. First, the theory is still justified away from the critical point, depending on the parameters. Secondly, it is qualitatively correct in many aspects and represents a crucial perspective from which one can understand the more correct approaches based on scaling and renormalization, which are also field theories, but differ in that they consider the effects of the ϕ^4 term.

7.5 Critical Phenomena, Scaling and Exponents

Phase transitions can be classified as either first or second order. First-order transitions are characterized by co-existing phases with the order parameter being different in the two phases. In second-order transitions the derivative (or perhaps n^{th} -order derivative) of the order parameter is discontinuous. The term ‘‘order parameter’’ refers to any measure, not even necessarily observable, used to describe a system in a given state. Examples are the magnetization density (ferromagnetic transition), number density (liquid-gas transition), quark-antiquark condensate (chiral transition),

Higg's condensate (electro-weak transition), or pair density (superconductivity). Quantities that involve comparing systems with different conditions, such as the specific heat, which involves comparing the entropy at two different temperatures, or the susceptibility, which involves comparing the magnetization at two different external fields, are not order parameters.

Transitions can belong to any of several universality classes, a grouping determined by the symmetry associated with the phase transition. For instance, in the $X - Y$ model the system breaks two-dimensional rotational symmetry by choosing a specific direction for aligning spins. The three-dimensional analog, the Heisenberg model, has a higher degree of symmetry, hence belongs to a different universality class and different critical exponents, which will be introduced further below.

At T_c Landau's field theory gave the following form for the correlation function for magnetization density in the previous section,

$$\Gamma(r) \equiv \langle m(\vec{r})m(r=0) \rangle \sim \frac{1}{r} e^{-r/\xi}, \quad \xi = \sqrt{\kappa/A}, \quad (55)$$

where the free energy density was treated in the Gaussian approximation,

$$f \approx \frac{A}{2} m^2 + \frac{\kappa}{2} (\nabla m)^2. \quad (56)$$

Here, A is a function of temperature. At T_c , the sign of A will switch, and the correlation length diverges as

$$\xi = \sqrt{\frac{\kappa}{dA/dT}} (T - T_c)^{-1/2} \sim t^{-1/2}, \quad t \equiv \frac{T - T_c}{T_c}. \quad (57)$$

In interacting theories, the effects of interactions change the behavior of ξ near T_c quantitatively, but not qualitatively. Quantum aspects of the interactions lead to screening effects that can be understood in renormalization calculations, but are beyond the scope of this class (See Chapter 18 in Huang). The resulting form for the correlation function near T_c is of the form,

$$\Gamma(r) \sim r^{2-d+\eta} e^{-r/\xi}, \quad \xi \sim t^{-\nu}. \quad (58)$$

Here, d is the number of dimensions. In the three-dimensional Landau calculation, $d = 3$, and the two *critical exponents* are

$$\eta = 0, \quad \nu = 1/2. \quad (59)$$

However, in the real world they vary from this amount, depending on the universality class.

The other critical exponents ($\alpha, \beta, \gamma, \delta$) are all determined by η and ν . The relations can best be understood from dimensional analysis. These arguments are based on two assumptions:

1. The non-analytic part of the free energy density scales proportional to $\sim 1/\xi^3$. From dimensional grounds, if ξ is the only relevant length scale, and since the free-energy density has dimensions of $1/L^3$, it must behave as ξ^{-3} .
2. The magnetization and fields all behave as if they have effective dimensionalities, such that changing t , which causes a scale change of ξ , will scale all quantities according to their effective dimensionality.

Following these arguments constrains all the other critical exponents as follows:

$$\langle \phi(0)\phi(r) \rangle \sim r^{2-d+\eta} e^{-r/\xi}, \quad \xi \sim t^{-\nu}, \quad \nu = 1/2, \quad (60)$$

$$C_v \sim t^{-\alpha}, \quad \alpha = 2 - \nu d, \quad (61)$$

$$\langle m \rangle (t=0) \sim H^{1/\delta}, \quad \delta = \frac{d - \eta + 2}{d + \eta - 2}, \quad (62)$$

$$\langle m \rangle (H=0, t < 0) \sim t^\beta, \quad \beta = \nu(d + \eta - 2)/2, \quad (63)$$

$$\chi \sim t^{-\gamma}, \quad \gamma = \nu(2 - \eta), \quad (64)$$

The two assumptions above were finally justified with renormalization theory. For this Ken Wilson was awarded the Nobel prize. Renormalization theory is beyond what we will cover in this class, but the interested student should read Chapter 18 of Huang.

7.6 Symmetry Breaking and Universality Classes

In Landau theory, the critical exponents were all determined by the dimensionality alone. However, in nature they are determined by the symmetry breaking involved in the problem. For instance, if there is a single real field ϕ whose free energy density is of the “mexican hat” form,

$$f = \frac{A}{2}\phi^2 + \frac{B}{4}\phi^4, \quad (65)$$

the system must choose whether to create a field with $\langle\phi\rangle > 0$ or < 0 when A becomes negative. For a complex field with a free energy density of the same form, the system must also choose the complex phase. For a system where the spins are allowed to align in any direction, the system must choose a direction (θ, ϕ) on which to align. Each of these three examples represents a different symmetry and leads to different exponents. The critical exponents are determined by the symmetry, and problems of the same symmetry are referred to as being in the same *universality class*. The breaking of the symmetry at low temperature is called *spontaneous symmetry breaking*, as opposed to the breaking of the symmetry by adding an external field, which is referred to as *explicit symmetry breaking*.

For the broken reflection symmetry above, the system chooses between two discrete minima, whereas for the example of the complex field or for the spin alignment, the set of possible minima form an infinite continuous set. These symmetries are called *continuous symmetries* and have the property that after being broken, one can make incremental changes in the field while maintaining a minimum in the free energy. For instance, consider the case of the complex field. Writing the free energy density in terms of the real and imaginary parts, ϕ_x and ϕ_y ,

$$f = \frac{A}{2}(\phi_x^2 + \phi_y^2) + \frac{B}{4}(\phi_x^2 + \phi_y^2)^2 + \frac{\kappa}{2}(\nabla\phi)^2, \quad (66)$$

where we consider the case where $\alpha < 0$ and minima appear for

$$(\phi_x^2 + \phi_y^2) = \phi_0^2 = -\frac{A}{B}. \quad (67)$$

If one expands around the minima in the x direction, one can rewrite the fields as

$$\delta\phi_x = \phi_x - \phi_0, \quad \delta\phi_y = \phi_y. \quad (68)$$

Rewriting the free energy in terms of $\delta\phi_x$ and $\delta\phi_y$,

$$\begin{aligned} f &\approx f_{\min} + \frac{1}{2}\delta\phi_x^2 \frac{\partial^2 f}{\partial^2 \phi_x} + \frac{1}{2}\delta\phi_y^2 \frac{\partial^2 f}{\partial^2 \phi_y} + \frac{\kappa}{2}(\nabla\phi)^2 \\ &= f_{\min} + \frac{1}{2}A_x\delta\phi_x^2 + \frac{1}{2}A_y\delta\phi_y^2 + \frac{\kappa}{2}(\nabla\delta\phi_x)^2 + \frac{\kappa}{2}(\nabla\delta\phi_y)^2, \\ A_x &= 4B/|\alpha|, \quad A_y = 0. \end{aligned} \quad (69)$$

Thus, the y mode contributes to the free energy only through the term proportional to $(\nabla\delta\phi_y)^2$, or in momentum space, the modes contribute to the free energy as k^2 . For relativistic field theories, modes for which the contribution vanishes as $k \rightarrow 0$ are called massless modes as they have no rest energy. These modes are also referred to as Goldstone bosons (Goldstone derived the general idea that massless modes should be associated with breaking a continuous symmetry) or Nambu-Goldstone bosons. Although the most famous example of such symmetry breaking

is the electro-weak transition in particle physics, Higgs, Goldstone and Nambu were concerned with the nuclear physics problem of chiral symmetry breaking, and of course Landau was most concerned with problems in condensed matter physics. It is no exaggeration to state that the most important problems addressed in the last 40 years in nuclear, particle and condensed matter physics center around spontaneously broken symmetries. Furthermore, one of the most exciting astronomical measurements, that of fluctuations of the 3-degree background radiation concerns inflationary theory, which might be associated with the breaking of GUT (grand unified theories) symmetries, under which quarks and leptons appear in the same multiplet as do all gauge bosons.

7.7 Problems

1. Consider the example for which the surface energy was calculated, where

$$\Delta\Psi \equiv P_0 - P + (\mu - \mu_0)\rho = \frac{A}{2} [(\rho - \rho_c)^2 - \alpha^2]^2.$$

Using Eq. (17), solve for the density profile $\rho(x)$ between the two phases.

2. Consider the one-dimensional Ising model, with the total energy in the mean field approximation being,

$$E = - \sum_i \frac{1}{2} q J \langle \sigma \rangle \sigma_i.$$

Note the factor of 1/2 being added relative to the effective energy for one mode to account for double counting.

- (a) Using the definition of entropy, $-\sum_i p_i \ln(p_i)$, show that the entropy per spin is:

$$S/N = -\frac{1 + \langle \sigma \rangle}{2} \ln \left(\frac{1 + \langle \sigma \rangle}{2} \right) - \frac{1 - \langle \sigma \rangle}{2} \ln \left(\frac{1 - \langle \sigma \rangle}{2} \right).$$

- (b) The free-energy, $F = E - TS$, per spin is then

$$F/N = -\frac{1}{2} q J \langle \sigma \rangle^2 + T \frac{(1 + \langle \sigma \rangle)}{2} \ln \left(\frac{1 + \langle \sigma \rangle}{2} \right) + T \frac{(1 - \langle \sigma \rangle)}{2} \ln \left(\frac{1 - \langle \sigma \rangle}{2} \right).$$

Show that minimizing the free energy w.r.t. $\langle \sigma \rangle$ gives:

$$2\beta q J \langle \sigma \rangle = \ln \left(\frac{(1 + \langle \sigma \rangle)}{(1 - \langle \sigma \rangle)} \right).$$

- (c) Compare the expression above to that from the previous expression,

$$\langle \sigma \rangle = \tanh(\beta q J \langle \sigma \rangle).$$

- (d) If the density of spin sites per unit volume is ρ_0 , the free energy density is

$$\begin{aligned} f(\sigma, T) &= \rho_0 \left\{ \mathcal{V}(\sigma, T) + \frac{\kappa}{2} (\nabla \sigma)^2 \right\}, \\ \mathcal{V}(\sigma, T) &= -\frac{1}{2} q J \sigma^2 + T \frac{(1 + \sigma)}{2} \ln \left(\frac{1 + \sigma}{2} \right) + T \frac{(1 - \sigma)}{2} \ln \left(\frac{1 - \sigma}{2} \right). \end{aligned}$$

Derive an expression for the surface free energy that is an integral over the density with limits from $-\sigma_{\text{eq}}$ and $+\sigma_{\text{eq}}$. Write the integrand in terms of ρ_0, κ, T and $\mathcal{R}(\sigma, T) \equiv \mathcal{V}(\sigma, T) - \mathcal{V}(\sigma_{\text{eq}}, T)$, where σ_{eq} is the solution to the transcendental expression for the equilibrated value for $\langle \sigma \rangle$.

(e) Find the surface energy in the limit that $T \rightarrow 0$ in terms of κ, ρ_0 and $T_c = qJ$.

3. Repeat the calculation for the correlation function from Sec. 7.3 for the one dimensional case. In this case, the Fourier transforms are defined as:

$$\begin{aligned}\tilde{\phi}_k &\equiv \frac{1}{\sqrt{L}} \int dx e^{ikx} \phi(x), \\ \phi(x) &= \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \phi_k.\end{aligned}$$

4. Beginning with the definition of the average density,

$$\langle \rho \rangle = \frac{\text{Tr} e^{-\beta H + \beta \mu \int d^3 r \rho(\vec{r})} \rho(\vec{r} = 0)}{\text{Tr} e^{-\beta H + \beta \mu \int d^3 r \rho(\vec{r})}},$$

prove the following relation between the density-density correlation function and $d\rho/d\mu$,

$$T \frac{d\langle \rho \rangle}{d\mu} = \int d^3 r \{ \langle \rho(\vec{r}) \rho(\vec{r} = 0) \rangle - \langle \rho \rangle^2 \}.$$

5. Returning to the expression for the mean spin in the Ising model in the mean field approximation,

$$\langle \sigma \rangle = \tanh(\beta q J \langle \sigma \rangle + \beta \mu B),$$

solve for $\langle \sigma \rangle$ in the limit that μB is small and $T = T_c$. Hint: Expand \tanh and keep only the lowest order non-zero terms for both $\langle \sigma \rangle$ and for μB .

6. Assume the following form for the correlation function,

$$\langle m(0)m(r) \rangle - \langle m \rangle^2 = C r^{2-d-\eta} e^{-r/\xi}, \quad \xi \sim t^{-\nu}$$

near T_c .

- (a) Find the susceptibility using Eq. (31) in terms of C, ξ, η and the dimensionality d . When expressing your answer, you can use the shorthand Ω_d for $\int d\Omega$ in d dimensions, i.e.,

$$\int d^d r \dots = \int d\Omega r^{d-1} dr \dots$$

For your entertainment, $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$.

- (b) If χ diverges as $t^{-\gamma}$ near T_c , find γ in terms of η and ν .

7. Assume that the free energy obeys the following form,

$$F = \int d^3 r \left\{ \frac{A}{2} \phi^2 + \frac{C}{2} \phi^6 \right\}.$$

Assuming that near $T_c, A \sim at$, find the critical exponent in mean field theory β where,

$$\langle \phi \rangle \sim t^\beta$$

below T_c .