Physics 831 Practice Quiz #4 - Wednesday, Nov. 28

YOUR NAME: _____________________

FUN FACTS TO KNOW AND TELL

\[ \int_0^\infty dx \, \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \quad \int_0^\infty dx \, \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n) \left[ 1 - (1/2)^{n-1} \right], \]

\[ \zeta(n) \equiv \sum_{m=1}^\infty \frac{1}{m^n}, \quad \Gamma(n) \equiv (n-1)!, \]

\[ \zeta(3/2) = 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90}, \]

\[ \int_{-\infty}^\infty dx \, e^{-x^2/2} = \sqrt{2\pi}, \quad \int_0^\infty dx \, x^n e^{-x} = n! \]

1. (2 pts each) Consider a single electron that can be either spin \( \uparrow \) or \( \downarrow \), with the two energies being \( \epsilon \) or \(-\epsilon\).

(a) What is the average energy when \( T = 0 \)?
(b) What is the average energy when \( T \to \infty \)?
(c) What is the specific heat when \( T = 0 \)?
(d) What is the specific heat when \( T \to \infty \)?
(e) What is the entropy when \( T = 0 \)?
(f) What is the entropy when \( T \to \infty \)?
2. (10 pts) Beginning with the fundamental thermodynamic relation, and the definition of the specific heat,

\[ TdS = dE + PdV - \mu dN, \quad C_V = T \left. \frac{\partial S}{\partial T} \right|_{N,V}, \]

derive the relation:

\[ -\frac{1}{T} \left. \frac{\partial C_V}{\partial N} \right|_{T,V} = \left. \frac{\partial^2 \mu}{\partial T^2} \right|_{V,T} \]
3. (10 pts) Consider a particle moving in a potential well:

\[ V(x) = -A \ln \left( \frac{x}{x_0} \right) + Bx, \quad \text{where } (x_0 > 0, A > 0, B > 0), \]

which confines a particle \( 0 < x < \infty \).

Find \( \langle x \rangle \) as a function of \( A, B, x_0 \) and the temperature \( T \).
4. Massless electrons ($\epsilon = pc$, and can be either spin up or spin down) move in **two dimensions** and equilibrate to a temperature $T$.

(a) (5pts) For $T = 0$ and chemical potential $\mu$, find the density (number per area) $\rho$ in terms of $\mu$.

(b) (5pts) What is $D(\epsilon)$, the density of single particle states?

(c) (5pts) To order $T^2$, find the change in the chemical potential $\delta \mu$ necessary to maintain constant density. Express answer in terms of $\mu$ and $T$. 