your name__________________________

Physics 831 Quiz #5 - Wednesday, Oct. 19, 2016

You may work in groups of three, but the group must contain nobody with whom you worked on a previous quiz. This quiz is open-note, open-book, open internet, and open-mind. However, you should not contact individuals outside your working group.

A large number of N diatomic molecules of mass m are confined to a region by a harmonic-oscillator potential,

$$V(\vec{r}) = \frac{1}{2}kr^2.$$

The system is at a sufficient temperature T so that the gas can be considered dilute. The temperature is in the range where rotational modes are routinely excited, but vibrational modes can be neglected.

1. (10 pts) What is the energy per particle? Give your answer in terms of m, T, and k.

2. (10 pts) Derive an expression for the entropy per particle in terms of the same variables. Begin with the expression,
$$S = \ln Z + \beta E,$$
where
$$Z = \frac{z^N}{N!},$$
and z is the partition function of a single molecule.

3. (10 pts) If the spring constant is adiabatically changed from k_i to k_f, and if the initial temperature is T_i, find T_f.

Fun facts to know and tell: $\lim_{N\to\infty} \ln(N!) = N\ln N - N$

Solutions

First calculate partition function of single particle

$$\begin{aligned} z &= \int \frac{d^3p d^3r}{(2\pi\hbar)^3} \exp\left\{-\beta\frac{p^2}{2m} - \beta\frac{kr^2}{2}\right\} \int d\ell(2\ell+1)e^{-\beta\hbar^2\ell(\ell+1)/2I}, \\ &\approx \frac{1}{\hbar^3}\left(\frac{mT^2}{k}\right)^{3/2}\frac{2IT}{\hbar^2}, \\ &= \frac{2I}{\hbar^5}\left(\frac{m}{k}\right)^{3/2} T^4. \end{aligned}$$

where the approximation is that $\beta\hbar^2/I << 1$, so that the rotational states can be treated like a continuum.

a) The energy per particle is

$$\begin{aligned} \frac{E}{N} &= -\partial_\beta \ln z \\ &= 4T \end{aligned}$$

b) The entropy is

$$\begin{aligned} \frac{S}{N} &= \ln z - \ln N + 1 + \beta \frac{\langle E \rangle}{N} \\ &= \ln\left[\left(\frac{m}{k}\right)^{3/2} \frac{2I}{N\hbar^5} T^4\right] + 5 \end{aligned}$$

c) Keep $T^4/k^{3/2}$ constant,

$$\begin{aligned} \frac{T_i^4}{k_i^{3/2}} &= \frac{T_f^4}{k_f^{3/2}} \\ T_f &= T_i \left(\frac{k_f}{k_i}\right)^{3/8}. \end{aligned}$$