your name(s)______________________________

Physics 831 Quiz #4
Friday, Sep. 29, 2017

Work in groups of three to four.

Eq. (3.1) from lecture notes:

$$P = \rho T \left[A_1 + \sum_{n=2}^{\infty} A_n \left(\frac{\rho}{\rho_0} \right)^{n-1} \right], \quad \rho_0 \equiv \frac{(2j+1)}{(2\pi\hbar)^3} \int d^3p \; e^{-\epsilon_p/T}. \tag{1}$$

Consider a low density two-dimensional gas of non-relativistic spin-s fermions of mass m at temperature $T = 1/\beta$ and chemical potential $\mu < 0$.

1. Show that $P = \rho T$ for $\mu << 0$. Begin with:

$$\begin{aligned} \rho &= (2s+1) \int \frac{d^2p}{(2\pi\hbar)^2} \; f(\vec{p}), \\ P &= (2s+1)T \int \frac{d^2p}{(2\pi\hbar)^2} \; \ln[1 + e^{-\beta(E-\mu)}], \\ f(\vec{p}) &= \frac{e^{-\beta(E-\mu)}}{1 + e^{-\beta(E-\mu)}}. \end{aligned}$$

   Here, P and ρ are the two-dimensional versions: P is a force per unit length and ρ is a number per unit area.

2. Find ρ_0 as defined in Eq. (1) in terms of m and T, but adjusting for two dimensions.

3. Expand the density ρ to second order in $e^{\beta\mu}$, i.e., to $e^{2\beta\mu}$. Express your answers for this part and the next two parts in terms of ρ_0.

4. Expand ρ^2 to second order in $e^{\beta\mu}$.

5. Expand $\delta P \equiv P - \rho T$ to second order in $e^{\beta\mu}$.

6. Determine the second virial coefficient defined by the two-dimensional version of Eq. (1)

1. $\ln(1+x) \sim x$, so as $e^{\beta\mu} \to 0$,

$$\rho = (2s+1) \int \frac{d^2p}{(2\pi\hbar)^2} e^{-\beta(\varepsilon-\mu)}$$

$$P = (2s+1) T \int \frac{d^2p}{(2\pi\hbar)^2} e^{-\beta(\varepsilon-\mu)}$$

$$= \rho T \quad \text{by inspection}$$

2.) $\rho_0 = (2S+1)\left(\int \frac{dp_x}{2\pi\hbar} e^{-p_x^2/2mT}\right)^2 = \left(\frac{\sqrt{mT}}{\sqrt{2\pi\hbar^2}}\right)^2 (2S+1)$

$= \frac{mT}{2\pi\hbar^2}(2S+1)$

3) $\rho \simeq (2S+1)\int \frac{d^2p}{(2\pi\hbar)^2}\left\{e^{\beta\mu}e^{-\beta E} - e^{2\beta\mu}e^{-2\beta E}\right\}$

$= \rho_0 e^{\beta\mu} - \rho_0 \cdot \frac{1}{2} e^{2\beta\mu} \frac{1}{(2S+1)}$

4) $\rho^2 \simeq \rho_0^2 e^{2\beta\mu}$

5) $P/T = \rho_0 e^{\beta\mu} - \int \frac{d^2p}{(2\pi\hbar)^2} \frac{1}{2} e^{2\beta\mu} e^{-2\beta E}$

$= \rho_0 e^{\beta\mu} - \frac{1}{4}\rho_0 e^{2\beta\mu}$

6) $P/T \simeq \rho + \frac{1}{4}\rho_0 e^{2\beta\mu} + \cdots$

From (4) $e^{2\beta\mu} \simeq \frac{\rho^2}{\rho_0^2}$

$\frac{P}{T} \simeq \rho + \frac{1}{4}\frac{\rho^2}{\rho_0^2}$

By comparison with Eq. (3.1)

$A_2 = \frac{1}{4}$