Statistical Mechanics
Review Problem

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1.

Show that the Liouville theorem,

$$\frac{d}{dt}\left[\Delta\Omega\right] = \frac{d}{dt}\left[\frac{1}{(2\pi\hbar)^3}\Pi_i \Delta x_i \Delta p_i\right] = 0,$$

in a fixed time step, follows from Hamilton's equations and the equality of mixed partials:

$$\frac{\partial H}{\partial p_i} = \frac{dx_i}{dt}$$
$$\frac{\partial H}{\partial x_i} = -\frac{dp_i}{dt}$$
$$\frac{\partial^2 H}{\partial x_i \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial x_i}$$

Starting with

$$\frac{\partial^2 H}{\partial x_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial x_i} = 0$$
$$\frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial x_i} = 0$$

Now using Hamilton's equations:

$$\frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right) - \frac{\partial}{\partial p_i}\left(-\frac{dp_i}{dt}\right) = 0$$
$$\Delta\left(\frac{dx_i}{dt}\right)\Delta p_i + \Delta\left(\frac{dp_i}{dt}\right)\Delta x_i = 0$$
$$\frac{d}{dt}\left(\Delta x_i \Delta p_i\right) = 0$$
$$\frac{d}{dt}\left((2\pi\hbar)^3 \Delta\Omega\right) = 0$$

2.

A beam of particles has a transverse width of 1 mm and its transverse momentum has a spread of 1 keV/c at its focal plane. Farther down the beamline, the width of the beam is measured to be 5 mm. Use Liouville's theorem to find the spread in momentum at this point on the beamline.

Since the phase space volume is constant, $\Delta x_T \Delta p_T$ is also constant. Therefore,

$$\begin{aligned}\Delta p'_T &= \Delta p_T \frac{\Delta x_T}{\Delta x'_T} \\ &= 1 \text{ keV/c}\,\frac{1\text{ mm}}{5\text{ mm}} \\ &= 200 \text{ eV/c}\end{aligned}$$