5.1 Practice Problem Solution

(a) Equate the number of uncoupled modes (2 modes per oscillator) to the number of modes you get from integrating all modes up to the cutoff wavenumber:

$$2N = \frac{2A}{(2\pi)^2}\int_0^{k_D} d^2k = \frac{2A}{(2\pi)^2}\cdot\left(2\pi\,\frac{k_D^2}{2}\right)$$

$$\rightarrow k_D = \sqrt{\frac{4\pi N}{A}} \tag{1}$$

This gives the Debye frequency,

$$\omega_D = c_s k_D = 2c_s\sqrt{\frac{\pi N}{A}} \tag{2}$$

(b) Total energy is proportional to the energy per momentum mode multiplied by the occupancy of that mode, integrated over all momenta up to the cutoff - in the limit $T << \hbar\omega_D$, this cutoff is infinity.

$$E = \frac{2A}{(2\pi\hbar)^2}\int_0^{\infty} d^2p\; pc_s\frac{e^{-\beta pc_s}}{1-e^{-\beta pc_s}} \tag{3}$$

Let $x = \beta pc_s$, so that $dx = \beta pc_s$ and

$$d^2p = 2\pi p\;dp = \frac{2\pi}{(\beta c_s)^2}x\;dx$$

This gives

$$\begin{aligned}
E &= \frac{4\pi AT^3}{(2\pi\hbar c_s)^2}\int_0^{\infty} dx\;x^2\frac{e^{-x}}{1-e^{-x}}\cdot\left(\frac{e^x}{e^x}\right)\\
&= \frac{4\pi AT^3}{(2\pi\hbar c_s)^2}\int_0^{\infty} dx\frac{x^2}{e^x-1}\\
&= \frac{4\pi AT^3}{(2\pi\hbar c_s)^2}\cdot\Gamma(3)\zeta(3)\\
&= \frac{2AT^3}{\pi\hbar^2c_s^2}\zeta(3)
\end{aligned} \tag{4}$$

Then

$$C(T << \hbar\omega_D) = \frac{1}{A}\frac{dE}{dT} = \frac{6T^2}{\pi\hbar^2 c_s^2}\zeta(3) \tag{5}$$

(c) As $T \to \infty$ coupling becomes negligible, and the specific heat equals the number of degrees of freedom.

$$C(T \to \infty) = 2N \tag{6}$$