Statistical Mechanics Subject Exam, Dec. 13, 2007

SECRET STUDENT NUMBER: STUDNUMBER

FUN FACTS TO KNOW AND TELL

$$\begin{aligned}
\int_0^\infty dx\ \frac{x^{n-1}}{e^x-1} &= \Gamma(n)\zeta(n), \quad \int_0^\infty dx\ \frac{x^{n-1}}{e^x+1} = \Gamma(n)\zeta(n)\left[1-(1/2)^{n-1}\right], \\
\zeta(n) &\equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \\
\zeta(3/2) &= 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90}, \\
&\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi}, \quad \int_0^\infty dx\ x^n e^{-x} = n!
\end{aligned}$$

LONG ANSWER SECTION

1. (15 pts) Beginning with the fundamental thermodynamic relation, and the definition for C_P,

$$TdS = dE + PdV - \mu dN, \quad C_P = T\ \left.\frac{\partial S}{\partial T}\right|_{N,P},$$

derive the equality

$$\left.\frac{\partial C_P}{\partial P}\right|_{T,N} = -T\ \left.\frac{\partial^2 V}{\partial T^2}\right|_{P,N}.$$

2. (15 pts) Consider the equation of state,

$$P(\rho, T) = \rho T e^{\rho/\rho_0} - a\frac{\rho^2}{\rho_0}.$$

Solve for the critical density ρ_c in terms of ρ_0 and a.

3. Consider a **two-dimensional** gas of spinless massless ($\epsilon = pc$) identical bosons.

   (a) (10 pts) If the chemical potential is zero, solve for the density of particles at temperature T.

   (b) (5pts) For a fixed finite density, can this gas form a Bose condensate below some critical temperature? Justify your answer.

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SHORT ANSWER SECTION

4. (2 pts each) Consider a single atom with spin $S = 3/2$ that couples to an external magnetic field such that the energy is, $H = -\mu\vec{B} \cdot \vec{S}$. Assume the fields are non-zero and positive, $\mu B > 0$.

   (a) What is the average energy when $T = 0$? ___________

   (b) What is the entropy when $T = 0$? ___________

   (c) What is the average energy when $T >> \mu B$? ___________

   (d) What is the entropy when $T >> \mu B$? ___________

   (e) What is the average spin projection $\langle S_z \rangle$ when $T >> \mu B$? ___________

5. (2 pts each) Consider a three-dimensional solid that acts like a metal for all temperatures.

   (a) As $T \to 0$, the specific heat will be dominated by: (circle one)
   PHONONS **ELECTRONIC EXCITATIONS** **PHOTONS**

   (b) As $T \to 0$, the specific heat will rise as $C \sim T^n$. What is n? ___________

   (c) What is C/N when $\epsilon_F >> T >> \hbar\omega_D$? ___________

6. (2 pts each) Initially, a volume $2V$ of low density gas is divided into two equal volumes of volume V by a thin partition, which is suddenly removed at time $t = 0$. Assume the gas on each side is equilibrated to a temperature T_0, and that the walls are well insulated from the outside world for $t > 0$. For each question the changes to the entropy and energy refer to the difference between $t = 0$ and a final time sufficient to re-equilibrate the two sides, but not so long that appreciable heat enters or leaves the partition.

   (a) (2 pts each) If the gas particles on the right and left are of the same species and same initial density, the entropy per particle will change by an amount: ___________, and the energy per particle will change by an amount: ___________.

   (b) If one side of the box is initially in a vacuum, the entropy per particle will change by an amount: ___________, and the energy per particle will change by an amount: ___________.

   (c) If the gas particles on the right and left are of different species, but of the same mass, and at the same initial density, the entropy per particle will change by an amount: ___________, and the energy per particle will change by an amount: ___________.

7. (3 pts) Two phase transitions will be of the same universality class if they have: (circle one)

   - The same order parameters
   - The same critical exponents
   - The same Goldstone bosons
   - The same dimensionality

8. (5 pts) Consider a free energy density $f(x, y, T)$, where x and y are order parameters. Write a form for the $T \to 0$ limit of the free energy density $f(x, y, T = 0)$ that would be a candidate for spontaneous breaking of a continuous symmetry:

____

9. (2 pts each) Consider a **one-dimensional** Ising model where each spin can have $\sigma = \pm 1$. The interactions in the system are all attractive, involve nearest neighbors only, and are given by, $H_{nn} = -J \sum_{ij \in n.n.} \sigma_i \sigma_j$. Each spin also experiences an interaction with an external field, $H_B = -\mu B \sum_i \sigma_i$, where $\mu B \geq 0$. For each question choose between:

   (a) $\langle \sigma \rangle \to -1$
   (b) $-1 < \langle \sigma \rangle < 0$
   (c) $\langle \sigma \rangle \to 0$
   (d) $0 < \langle \sigma \rangle < 1$
   (e) $\langle \sigma \rangle \to 1$

   A) For $T \to \infty$ and $B = 0$: ____________.
   B) For $T \to 0$ and $B = 0$: ____________.
   C) For $T \to \infty$ and $B =$constant: ____________.
   D) For $T \to 0$ and $B =$constant: ____________.
   E) For $T \to$ constant and $B =$constant: ____________.

10. (5pts) Consider a **three-dimensional** Ising model, where each spin can have $\sigma = \pm 1$, and nearest neighbor spins experience an attractive interaction, $H_{nn} = -J \sum_{ij} \sigma_i \sigma_j$. Each spin also experiences an interaction with an external field, $H_B = -\mu B \sum_i \sigma_i$, where $\mu B > 0$. For a **constant non-zero field**, make a plot (qualitatively) to illustrate the behavior of $\langle \sigma \rangle$ vs. T in the space below: