SECRET STUDENT NUMBER: STUDNUMBER

FUN FACTS TO KNOW AND TELL

$$\int_0^\infty dx\, \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\, \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n)\left[1 - (1/2)^{n-1}\right],$$

$$\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!,$$

$$\zeta(3/2) = 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90},$$

$$\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\, x^n e^{-x} = n!$$

LONG ANSWER SECTION

1. (10 pts) Beginning with the fundamental thermodynamic relation,

$$TdS = dE + PdV - \mu dN,,$$

derive the equality

$$\left.\frac{\partial \mu}{\partial P}\right|_{S,N} = \left.\frac{\partial V}{\partial N}\right|_{S,P}.$$

2. Consider a one-dimensional array of N coupled harmonic oscillators in length L. The oscillator's movement is also confined to one dimension. In the absence of the coupling, each oscillator has a fundmamental frequency ω_0. Compression modes are characterized by a speed of sound c_s.

   (a) (10 pts) Solve for the Debye frequency, ω_D, in terms of ω_0, N, L and c_s.

   (b) (10 pts) For $T << \omega_D$, find the specific heat,

   $$C \equiv (1/L)d\langle E\rangle/dT.$$

   (c) (5 pts) What is $C(T \to \infty)$?

Extra work space for #2

3. (15 pts) The partition function for a one-dimensional spin system in a magnetic field B is

$$Z = \mathrm{Tr}\exp\left(-\beta H_0 + \beta\mu_b B \int dx\ m(x)\right).$$

After some calculations, one finds the spin-spin correlation function for zero field to be,

$$\langle (m(x=0) - \langle m\rangle)(m(x) - \langle m\rangle)\rangle = A(T)e^{-|x|/\ell}.$$

Calculate the susceptibility,

$$\chi \equiv \frac{d\langle m\rangle}{dB},$$

and express the answer in terms of T, A, ℓ and μ_b.

Extra work space for #3

SECRET STUDENT NUMBER: STUDNUMBER

SHORT ANSWER SECTION

4. (2 pts each) Two identical spin-zero particles can each occupy one of three single-particle energy levels, $-\epsilon$, 0 and ϵ.
   (a) What is the average energy when $T = 0$? ____________
   (b) What is the entropy when $T = 0$? ____________
   (c) What is the average energy when $T >> \epsilon$? ____________
   (d) What is the entropy when $T >> \epsilon$? ____________

5. (3 pts each) Consider a non-interacting Fermi gas of a fixed number of particles in a fixed volume (or fixed area or length in 2-d or 1-d). As $T \to 0$, the excitation energy rises as $E^* \sim T^\nu$.
   (a) If the system is one-dimensional, what is ν? ____________
   (b) If the system is three-dimensional, what is ν? ____________

6. (4 pts) Imagine a box with volume V in which there is a dilute gas of N ^{4}He atoms. The volume in the box is doubled and the number of ^{4}He atoms particles is also increased by a factor of 2, while keeping the temperature fixed. Consider the limit of large N and V. The entropy per particle S/N: (Circle one)
   (a) decreases
   (b) stays the same
   (c) increases

7. (4 pts) Consider two dilute diatomic gases at low temperature, but not so low as to ignore rotational excitations completely. (Consider the gases sufficiently dilute so that you can ignore quantum statistics and inter-molecular interactions). In gas A, each molecule is comprised of two ^{16}O atoms, and in gas B, each molecule is comprised of two ^{18}O atoms. Which gas has the higher specific heat?
   (a) A
   (b) B
   (c) they are the same

8. (4 pts) One might expect a Goldstone boson from a phase transition with: (circle one)
   (a) spontaneous breaking of a continuous symmetry
   (b) spontaneous breaking of a discrete symmetry
   (c) explicit breaking of a continuous symmetry
   (d) explicit breaking of a discrete symmetry

9. (4 pts) If you read an article where the authors maximize the pressure to solve for an order parameter, which quantities can you assume were fixed in the calculation? (Circle all that apply)

   (a) entropy

   (b) volume

   (c) temperature

   (d) particle number

   (e) density

   (f) chemical potential

   (g) energy density

10. (2 pts each) Graph several isotherms on a P vs. V graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

   (a) An isotherm with $T > T_c$.

   (b) An isotherm with $T = T_c$.

   (c) An isotherm with $T < T_c$.

   (d) Label the critical point.

   (e) For the isotherm with $T < T_c$, label the coexistence points.