Statistical Mechanics Subject Exam, Dec. 11, 2013

SECRET STUDENT NUMBER: STUDNUMBER

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

FUN FACTS TO KNOW AND TELL

$$\begin{aligned}
\int_0^\infty dx\ \frac{x^{n-1}}{e^x-1} &= \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\ \frac{x^{n-1}}{e^x+1} = \Gamma(n)\zeta(n)\left[1-(1/2)^{n-1}\right], \\
\zeta(n) &\equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \\
\zeta(3/2) &= 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90}, \\
& \int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi}, \quad \int_0^\infty dx\ x^n e^{-x} = n!
\end{aligned}$$

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LONG ANSWER SECTION

1. (10 pts) Consider two single-particle energy levels, 0 and ϵ. Spin-1 bosons ($m = -1, 0, 1$) are allowed to populate the levels and equilibrate with a heat and particle bath defined by a temperature T and chemical potential $\mu < 0$. The bosons are indistinguishable aside from their spin. What is the average number of bosons in each level?

Extra workspace for #1

2. (10 pts) Assume that the free energy in a two-dimensional system obeys the following form,

$$F = \int d^2r \left\{ \frac{A}{2}\phi^2 + \frac{C}{2}\phi^6 \right\}.$$

Assuming that near T_c, $A \sim at$, find the critical exponent in mean field theory β where,

$$\langle \phi \rangle \sim t^\beta$$

below T_c.

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Extra workspace for #2

3. N ink molecules are placed in a liquid at a time $t = 0$ and diffuse according to a diffusion constant D, i.e., the density of molecules satisfies the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}.$$

For example, if the N molecules are initially positioned at $x = 0$ in a translationally-invariant medium, the density evolves as,

$$\rho(x, t) = \frac{N}{\sqrt{4\pi D t}} e^{-x^2/4Dt}.$$

(a) (10 pts) Now, add an absorptive boundary at $x = 0$, and place the drop at a small distance a from the boundary. By *small* we will only consider times such that $2Dt >> a^2$. Solve for the density $\rho(x, t)$. You should include only the lowest order in a.

(b) (5 pts) What fraction of molecules survive to time t? Again assume $2Dt >> a^2$.

Extra workspace for #3

4. Suppose the average energy $\bar{E}$ and the average number of particles $\bar{N}$ in a one-dimensional system of extent L are given as a function of T, L and $\alpha \equiv -\mu/T$. Further assume that L is much larger than any microscopic scale or correlation length of the system.

   (a) (10 pts) Derive an expression for the specific heat per unit length,

   $$C \equiv \frac{1}{L}\frac{\partial \bar{E}}{\partial T}\bigg|_N,$$

   in terms of $T, L, \bar{E}, \bar{N}, \partial_T\bar{E}|_\alpha, \partial_\alpha\bar{E}|_T$ and $\partial_\alpha\bar{N}|_T$.

   (b) (10 pts) Assume the correlations in the system are sufficiently local they can be expressed in terms of delta functions,

   $$\begin{aligned}\langle\Delta\rho(0)\Delta\rho(x)\rangle|_{\alpha,T} &= A\delta(x),\\ \langle\Delta\epsilon(0)\Delta\epsilon(x)\rangle|_{\alpha,T} &= B\delta(x),\\ \langle\Delta\epsilon(0)\Delta\rho(x)\rangle|_{\alpha,T} &= D\delta(x),\end{aligned}$$

   where ϵ and ρ are the energy density and number density respectively. Express C in terms of T, α, A, B and D.

Extra work space for #4

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SHORT ANSWER SECTION

5. (1 pt each) Graph several isotherms on a P vs. V graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

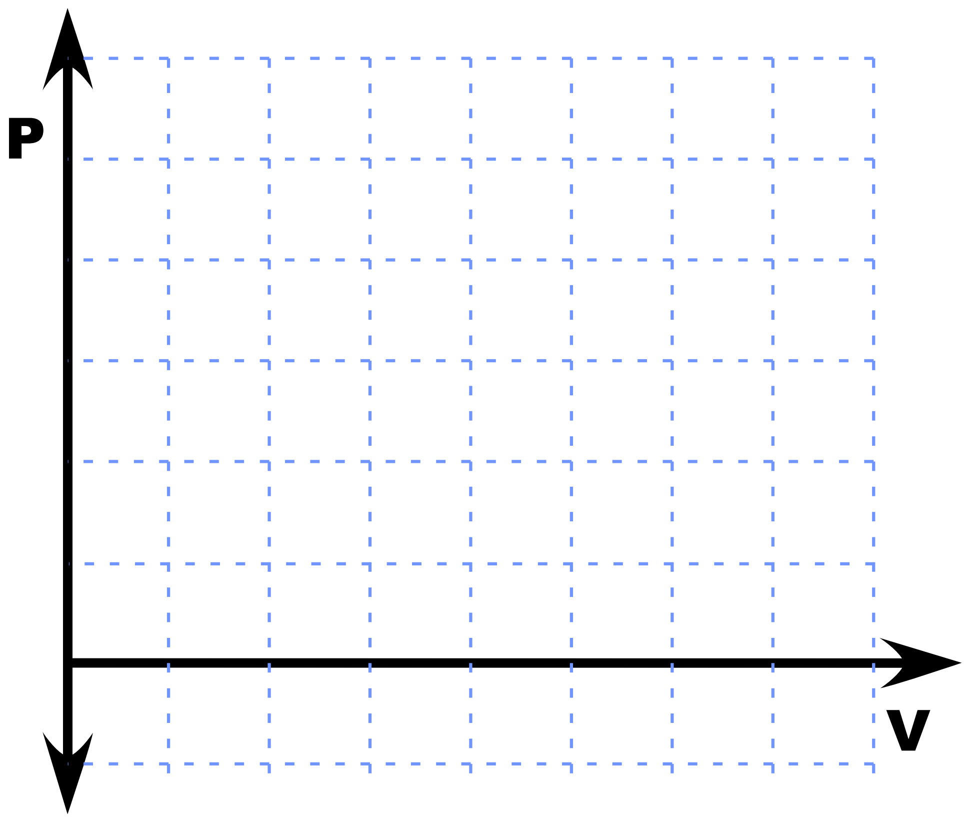
   (a) An isotherm with $T > T_c$.

   (b) An isotherm with $T = T_c$.

   (c) An isotherm with $T < T_c$.

   (d) Label the critical point.

   (e) For the isotherm with $T < T_c$, label the coexistence points.



6. (2 pts each) Consider a one-dimensional Ising model. Label each of the following statements as true or false.

   (a) In the exact solution there is no phase transition. ________

   (b) In the mean-field solution there is no phase transition. ________

   (c) In the mean-field solution, the critical exponents are the same for the one-dimensional and two-dimensional solutions. ________

7. (3 pts) A solution of fixed number is kept at fixed volume and room temperature in a tightly closed flask. When it adjusts its chemical compositions to approach thermodynamic equilibrium, which of the below is true? (choose the single most correct answer)

   (a) The net Gibbs free energy of the solution is minimized

   (b) The net entropy of the solution is maximized

   (c) The net Helmholtz free energy of the solution is minimized

   (d) (a) and (b)

   (e) all of the above

8. (2 pts each) Consider a simple model of a solid as being a lattice of ions that carry phonons, and a non-interacting electron gas. As the temperature goes to zero, which contribution dominates the specific heat as the temperature goes to zero? For each case below, enter *phonons, electrons* or *neither* if both contributions are of similar strength. If both contributions behave with the same power of T at low temperature, you can safely assume that the Fermi velocity is much greater than the speed of sound, $v_F >> c_s$.

   (a) For a one-dimensional crystal, ________

   (b) For a two-dimensional crystal, ________

   (c) For a three-dimensional crystal, ________

9. (3 pts) One might expect a Goldstone boson from a phase transition with: (circle one)
   - spontaneous breaking of a continuous symmetry
   - spontaneous breaking of a discrete symmetry
   - explicit breaking of a continuous symmetry
   - explicit breaking of a discrete symmetry
10. (2 pts) For a system of massless bosons, $E = cp$, what dimensionality, D, is required for Bose condensation?

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