your name(s)____

Physics 841 Quiz #5 - Monday, Feb. 27

You may work in groups of 3 (no more than one person from previous group)

Open note, open book, open internet, open mouth...

Consider a local group of charges around the origin in a two-dimensional world. The potential due to a point charge Q at position $x = a, y = 0$, as measured by an observer at coordinate r, ϕ is

$$\begin{aligned}
\Phi(r,\phi) &= -Q \ln\left(\sqrt{r^2 - 2ra\cos\phi + a^2}\right) \\
&= Q\left\{-\ln(r) + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{a}{r}\right)^n \cos n\phi\right\}. \qquad (1)
\end{aligned}$$

This expression assumes $r > a$.

1. Consider an areal charge density of charge, σ, that is completely contained within some radius R of the origin. The potential observed for $r > R$ can be written as an expansion,

$$\Phi(r,\phi) = A_0 \ln(r) + \sum_{n=1}^{\infty} \frac{A_n}{r^n}\cos n\phi + \frac{B_n}{r^n}\sin n\phi.$$

   Express the coefficients A_n and B_n in terms of $\sigma(\vec{r})$.

2. Find A_n and B_n for a single charge Q at $x = a, y = 0$. Then show that the potential indeed sums to $-Q\ln(r - a)$ when $\vec{r}$ is on the x axis. Do not use the identity in Eq. (1).

Solutions:

1.

$$\begin{aligned}
\Phi(r,\phi) &= \sum_i q_i'\left\{-\ln(r) + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_i'}{r}\right)^n \cos n(\phi - \phi_i')\right\} \\
&= \int d^2r'\ \sigma(\vec{r}')\left\{-\ln(r) + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r'}{r}\right)^n \cos n(\phi-\phi')\right\}, \\
&= \int d^2r'\ \sigma(\vec{r}')\left\{-\ln(r) + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r'}{r}\right)^n (\cos n\phi\cos n\phi' + \sin n\phi \sin n\phi')\right\}, \\
A_0 &= -\int d^2r'\ \sigma(\vec{r}'), \\
A_{n>0} &= \frac{1}{n}\int d^2r'\ \sigma(\vec{r}')r'^n \cos n\phi', \\
B_{n>0} &= \frac{1}{n}\int d^2r'\ \sigma(\vec{r}')r'^n \sin n\phi'.
\end{aligned}$$

2.

$$\begin{aligned}
A_0 &= -Q, \\
A_{n>0} &= Qa^n/n, \\
B_n &= 0, \\
\Phi(r, \phi = 0) &= Q\left\{ -\ln(r) + \sum_{n>0} \frac{1}{n}\frac{a^n}{r^n} \right\}, \\
\ln(r-a) &= \ln(r(1-a/r)) \\
&= \ln(r) + \ln(1-a/r) \\
&= \ln(r) + \sum_{n>0} \frac{(-1)^{n+1}}{n}\left(\frac{-a}{r}\right)^n \\
&= \ln(r) - \sum_{n>0} \frac{1}{n}\left(\frac{a}{r}\right)^n, \\
\Phi(r, \phi = 0) &= -Q\ln(r-a). \qquad \checkmark
\end{aligned}$$