FINAL EXAM,

PHYSICS 851, FALL 2019

Thursday, December 12, 7:45-9:45 AM

SECRET STUDENT NUMBER: <u>STUDNUMBER</u>

This exam is worth 150 points, but graded on a scale of 135 points.

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$

$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P}-e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y),$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B]=C,\ \text{ and } [C,A]=[C,B]=0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,$$

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle.$$

1. A neutron and proton occupy the ground state of a harmonic oscillator. The particles then feel two additional sources of interaction. First, they have a spin-spin interaction,

$$V_{\text{s.s.}} = \alpha \vec{S}^{(\text{n})} \cdot \vec{S}^{(\text{p})},$$

and secondly, they experience an external magnetic field

$$V_B = -\left(\mu_n \vec{S}^{(\text{n})} + \mu_p \vec{S}^{(\text{p})}\right) \cdot \vec{B}.$$

Letting J and M reference the total angular momentum and its projection, and letting m_n and m_s reference the projections of the neutron and protons spins,

(a) (10 pts) circle the operators that commute with the Hamiltonian,

- The magnitude of the total angular momentum, $|\vec{J}|^2$.
- J_z
- $S_z^{(n)}$
- $S_z^{(p)}$
- $|\vec{S}^{(p)}|^2$

(b) (10 pts) In the J, M basis,

$$|J=1, M=1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ |J=1, M=-1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|J=1, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |J=0, M=0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

write the Hamiltonian as a 4×4 matrix.

(c) (5 pts) Find the eigen-energies of the Hamiltonian.

(Extra work space for #1)

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2. A particle of mass m scatters off a target with a spherically symmetric potential,

$$V(r) = V_0 \Theta(R - r).$$

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the momentum p, where the energy is less than V_0.

(b) (5 pts) What is the cross-section in the limit that $p \rightarrow 0$?

(Extra work space for #2)

3. (15 pts) A particle is in the ground state of a two-level system where the Hamiltonian is

$$\boldsymbol{H_0 = V_0 \sigma_z.}$$

An interaction is then added,

$$\boldsymbol{V(t) = \Theta(t) \beta \sigma_x.}$$

What is the expectation of $\boldsymbol{\sigma_z}$ as a function of time?

(Extra work space for #3)

4. In one dimension, a particle of type a and mass m is in the ground state of an attractive potential

$$V_0(x) = -\beta\delta(x).$$

A perturbative potential is added,

$$V_{ab} = \alpha\cos(\omega t),$$

where α is small and $\hbar\omega$ is larger than the binding energy. This converts the particle to a type b particle, which has the same mass m but does not feel the effects of V_0.

(a) (10 pts) What is the binding energy of the a particle?

(b) (20 pts) What is the decay rate?

(Extra work space for #4)

5. (20 pts) The cross section for scattering of a particle with momentum $\hbar \boldsymbol{k}$ off a single target is

$$\frac{d\sigma}{d\Omega} = \alpha(\cos^2\theta + 1/5),$$

Now, two targets are placed a distance a apart, separated along the z axis (the same axis as the incident beam moves). At what scattering angles, θ, does the differential cross section, $d\sigma/d\Omega$, vanish?

(Extra work space for #5)

6. (20 pts) A particle of mass m is in an attractive Coulomb potential, $V = -e^2/r$. Using a Gaussian form,

$$\psi = e^{-r^2/2a^2},$$

as a trial form for the ground state wave function. Provide a variational estimate (upper-bound) for the ground state binding energy.

(Extra work space for #6)

7. A positively charged particle of mass m and charge e is placed in a region with uniform magnetic field B along the z axis.

   (a) (5 pts) Write the vector potential that describes the magnetic field such that $\vec{A}$ is in the $\hat{y}$ direction.

   (b) (5 pts) What is the lowest eigen-energy?

   (c) (5 pts) What is a general form for all the eigen-energies?

   (d) (10 pts) A uniform electric field, $E < B$, is then added in the x direction. If the particle is initially at $x = y = 0$ at time $t = 0$ and if the initial velocity is small, find its average velocity after a long time t. By "average", ignore any oscillatory forms to its position vs time. For full credit, derive the answer, for half credit, just write it down.

(Extra work space for #7)