MIDTERM EXAM,
PHYSICS 852, Spring 2020
Friday, Feb. 28, 1:50-2:40 PM
This exam is worth 60 quiz points

SECRET STUDENT NUMBER:
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$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$
$$H = i\hbar\partial_t,\ \vec{P} = -i\hbar\nabla,$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\ V(t')U(t',-\infty),$$
$$\langle x|x'\rangle = \delta(x-x'),\ \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$
$$|p\rangle = \int dx\ |x\rangle e^{ipx/\hbar},\quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$
$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$
$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\quad b^2 = \frac{\hbar}{m\omega},$$
$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$
$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$
$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$
$$\text{For } V = \beta\delta(x-y):\quad -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$
$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A},\quad \vec{B} = \nabla\times\vec{A},$$
$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$
$$e^{A+B} = e^Ae^Be^{-C/2},\ \text{ if } [A,B] = C,\ \text{ and } [C,A] = [C,B] = 0,$$
$$Y_{0,0} = \frac{1}{\sqrt{4\pi}},\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$
$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1),\quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi},$$
$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm2i\phi},\quad Y_{\ell-m}(\theta,\phi) = (-1)^mY^*_{\ell m}(\theta,\phi).$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x},\ j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{single}}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1) e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k} P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,\quad \sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

$$L_\pm|\ell, m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle,$$

$$C^{\ell,s}_{m_\ell,m_s;JM} = \langle \ell, s, J, M|\ell, s, m_\ell, m_s\rangle,$$

$$\langle\tilde{\beta}, J, M|T^k_q|\beta, \ell, m_\ell\rangle = C^{k\ell}_{qm_\ell;JM}\frac{\langle\tilde{\beta}, J||T^{(k)}||\beta, \ell, J\rangle}{\sqrt{2J+1}},$$

$$n = \frac{(2s+1)}{(2\pi)^d}\int_{k<k_f} d^dk,\quad d \text{ dimensions},$$

$$\{\Psi_s(\vec{x}), \Psi^\dagger_{s'}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_{ss'},$$

$$\Psi^\dagger_s(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} a^\dagger_s(\vec{k}),\quad \{\Psi_s(\vec{x}), a^\dagger_\alpha\} = \phi_{\alpha,s}(\vec{x}).$$

1. Consider a one-dimensional world where a type-A particle of mass m is confined by a harmonic oscillator potential,

$$V_a(x) = \frac{1}{2} m\omega^2 x^2.$$

The particle can decay to a type-B particle of the same mass, but the type-B particle does not feel the potential. The interaction responsible for the decay is

$$H_{\rm int} = g \int dx \, \left[\Psi_A^\dagger(x)\Psi_B(x) + \Psi_B^\dagger(x)\Psi_A(x)\right],$$

$$\Psi_A^\dagger(x) = \frac{1}{\sqrt{L}}\sum_k e^{-ikx}a_k^\dagger, \quad \Psi_B^\dagger(x) = \frac{1}{\sqrt{L}}\sum_k e^{-ikx}b_k^\dagger.$$

(a) (10 pts) Calculate the matrix element $\langle k_f|H_{\rm int}|i\rangle$, where k_f is the momentum of the outgoing type-B particle and i refers to the initial state of the type-A particle, which is in the ground state of the harmonic oscillator.

(b) (15 pts) Calculate the rate at which the type-A particle decays into a type-B particle.

(a) $$\langle f|H_{int}|i\rangle = \int g\langle k_f|\Psi_B^+(x)|0\rangle\langle 0|\Psi_A(x)|n=0\rangle dx$$

$$= \frac{g}{L^{1/2}}\int dx\, e^{-ik_f x} e^{-x^2/2b^2}\frac{1}{(\pi b^2)^{1/4}}$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$= \frac{g}{L^{1/2}} e^{-k_f^2 x^2/2}\frac{1}{(\pi b^2)^{1/4}}\int dx\, e^{-\frac{(x-ik_f b^2)^2}{2b^2}}$$

$$= \frac{g}{L^{1/2}} e^{-k_f^2 b^2/2}\frac{1}{(\pi b^2)^{1/4}}(2\pi b^2)^{1/2} = \frac{g}{L^{1/2}}e^{-k_f^2 b^2/2}(4\pi b^2)^{1/4}$$

(b) $$\Gamma = \frac{2\pi}{\hbar}\sum_k \frac{g^2(4\pi b^2)^{1/2}}{L} e^{-k_f^2 x^2}\,\delta(E_f - \hbar\omega/2)$$

$$= \frac{2\pi}{\hbar} g^2\frac{(4\pi b^2)^{1/2}}{\pi}\int_0^\infty dk\, \delta\left(\frac{\hbar^2 k^2}{2m} - \hbar\omega/2\right) e^{-k_f^2 b^2}$$

$$= \frac{4g^2}{\hbar} b\pi^{1/2}\frac{m}{\hbar^2 k_f} e^{-k_f^2 b^2}, \quad k_f = \sqrt{\frac{m\omega}{\hbar}} = \frac{1}{b}$$

$$= \frac{4g^2}{\hbar^3} m\frac{\hbar}{m\omega}\pi^{1/2} e^{-k^2b^2} = \frac{4g^2}{\hbar^2\omega}\pi^{1/2}e^{-k^2b^2}$$

$$= \frac{4g^2\pi^{1/2}}{\hbar^2\omega}e^{-1}$$

(Extra work space for #1)

2. Consider the following matrix element,

$$\boldsymbol{\mathcal{M}_{m_i m_f} = \langle \alpha, \ell_f = 1, m_f | P_z | \beta, \ell_i = 2, m_i \rangle}.$$

(a) (10 pts) For which combinations of $\boldsymbol{m_i, m_f}$ is $\boldsymbol{\mathcal{M}_{m_i m_f}}$ non-zero?

(b) (15 pts) If one were to calculate the matrix element

$$\boldsymbol{\mathcal{M}_{00} = \langle \alpha, \ell_f = 1, m_f = 0 | P_z | \beta, \ell_i = 2, m_i = 0 \rangle},$$

express all the non-zero elements of the operator $\boldsymbol{P_x}$,

$$\boldsymbol{\langle \alpha, \ell_f = 1, m_f | P_x | \beta, \ell_i = 2, m_i \rangle},$$

in terms of $\boldsymbol{\mathcal{M}_{00}}$ and Clebsch-Gordan coefficients. You can leave your answer in terms of Clebsch-Gordan coefficients. In fact DO NOT evaluate the Clebsch Gordan coefficients.

a

m_i	m_f
1	1
0	0
-1	-1

← non-zero

b $T^1_0 = P_z \qquad T^1_{\pm 1} = \mp (P_x \pm i P_y) \frac{1}{\sqrt{2}}$

$$P_x = \left(T^1_{-1} - T^1_1\right)\frac{1}{\sqrt{2}}$$

m_i	m_f	
2	1	$\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{-12;11}}{C^{12}_{00;10}}$
1	0	$\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{-11;10}}{C^{12}_{00;10}}$
0	1	$-\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{10;11}}{C^{12}_{00;10}}$
0	-1	$\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{-10;1-1}}{C^{12}_{00;10}}$
-1	0	$-\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{1-1;10}}{C^{12}_{00;10}}$
-2	-1	$-\frac{1}{\sqrt{2}} M_{00} \frac{C^{12}_{1-2;1-1}}{C^{12}_{00;10}}$

(Extra work space for #2)

3. Consider a zero-temperature non-interacting quark-gas (up and down quarks) which is also accompanied by electrons to balance the electric charge. The three species have Fermi momenta, $\hbar k_u, \hbar k_d$ and $\hbar k_e$. Assume all particles have ZERO MASS, which is not a bad assumption for very high densities. Normally, the density of a Fermi gas would be

$$\rho = \frac{2s+1}{6\pi^2}k_f^3,$$

but for quarks there are also three colors for each of the two spins, so the degeneracy factor $(2s+1) \rightarrow 6$. The electric charges of the three species are $q_u = 2e/3, q_d = -e/3, q_e = -e$. The baryon charge of either species of quarks is $1/3$, whereas electrons carry no baryon density. The weak interaction,

$$u + e \leftrightarrow d + \nu_e,$$

proceeds in such a way as to minimize the energy for a fixed baryon density, ρ_B, with the neutrinos being ignored because as massless neutral particles they can exit the system at will.

(a) (7 pts) Express ρ_B in terms of the Fermi wave numbers k_u, k_d and k_e.

(b) (7 pts) Express electric charge conservation in terms of the Fermi wave numbers.

(c) (7 pts) In terms of the Fermi wave numbers write an equation that expresses the fact that the overall energy is minimized.

(d) (4 pts) (no equations) Describe how you would go about finding the densities of each species given ρ_B.

a $\rho_B = \frac{6}{6\pi^2}\left(k_u^3 + k_d^3\right)$

b $0 = \frac{2}{6\pi^2}\left(2k_u^3 - k_d^3 - k_e^3\right)$

c $k_u + k_e = k_d$

d Either: I. Do 3-D Newton's method to solve 3 eq.'s / 3 unknowns

II. Plug / Substitute to reduce to 1 eq. 1 unknown, then solve with Newton's method.

This gives k_u, k_d, k_e

$\rho_u = \frac{1}{\pi^2}k_u^3,\ \rho_d = \frac{1}{\pi^2}k_d^3,\ \rho_e = \frac{1}{3\pi^2}k_e^3$

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(Extra work space for #3)