

MIDTERM PRACTICE EXAM,

PHYSICS 852, Spring 2020

Friday, Feb. 28, 1:50-2:40 PM

This practice exam is worth 0 points

SECRET STUDENT NUMBER:

STUDNUMBER

$$\int_{-\infty}^{\infty} dx e^{-x^2/2} = \sqrt{2\pi},$$

$$H = i\hbar\partial_t, \quad \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t, -\infty) = 1 + \frac{-i}{\hbar} \int_{-\infty}^t dt' V(t')U(t', -\infty),$$

$$\langle x|x'\rangle = \delta(x - x'), \quad \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p - p'),$$

$$|p\rangle = \int dx |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar} |p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X - i\sqrt{\frac{1}{2\hbar m\omega}} P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}} e^{-x^2/2b^2}, \quad b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r}, t) = \psi^*(\vec{r}_1, t_1)\psi(\vec{r}_2, t_2)$$

$$\vec{j}(\vec{r}, t) = \frac{-i\hbar}{2m} (\psi^*(\vec{r}, t)\nabla\psi(\vec{r}, t) - (\nabla\psi^*(\vec{r}, t))\psi(\vec{r}, t))$$

$$- \frac{e\vec{A}}{mc} |\psi(\vec{r}, t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

For $V = \beta\delta(x - y)$: $-\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon} \right) = -\beta\psi(y),$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}, \quad \vec{B} = \nabla \times \vec{A},$$

$$\omega_{\text{cyclotron}} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2}, \quad \text{if } [A, B] = C, \text{ and } [C, A] = [C, B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi},$$

$$Y_{2,0} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1), \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{\pm i\phi},$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\phi}, \quad Y_{\ell-m}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi).$$

$$|N\rangle = |n\rangle - \sum_{m \neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \dots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m \neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x}, \quad n_0(x) = -\frac{\cos x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3}{x^2} \cos x, \quad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right) \cos x - \frac{3}{x^2} \sin x,$$

$$\frac{d}{dt} P_{i \rightarrow n}(t) = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2 \hbar^4} \left| \int d^3r \mathcal{V}(r) e^{i(\vec{k}_f - \vec{k}_i) \cdot \vec{r}} \right|^2,$$

$$\sigma = \frac{(2S_R + 1) 4\pi (\hbar\Gamma_R/2)^2}{(2S_1 + 1)(2S_2 + 1) k^2 (\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{single}} \tilde{S}(\vec{q}), \quad \tilde{S}(\vec{q}) = \left| \sum_{\delta\vec{a}} e^{i\vec{q} \cdot \delta\vec{a}} \right|^2,$$

$$e^{i\vec{k} \cdot \vec{r}} = \sum_{\ell} (2\ell + 1) i^{\ell} j_{\ell}(kr) P_{\ell}(\cos \theta),$$

$$P_{\ell}(\cos \theta) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell, m=0}(\theta, \phi),$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/3,$$

$$f(\Omega) \equiv \sum_{\ell} (2\ell + 1) e^{i\delta_{\ell}} \sin \delta_{\ell} \frac{1}{k} P_{\ell}(\cos \theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R \rightarrow \infty} = e^{i\vec{k} \cdot \vec{r}} + \frac{e^{ikr}}{r} f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2, \quad \sigma = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \sin^2 \delta_{\ell},$$

$$L_{\pm} |\ell, m\rangle = \sqrt{\ell(\ell + 1) - m(m \pm 1)} |\ell, m \pm 1\rangle,$$

$$C_{m_{\ell}, m_s; JM}^{\ell, s} = \langle \ell, s, J, M | \ell, s, m_{\ell}, m_s \rangle,$$

$$\langle \tilde{\beta}, J, M | T_q^k | \beta, \ell, m_{\ell} \rangle = C_{qm_{\ell}; JM}^{k\ell} \frac{\langle \tilde{\beta}, J || T^{(k)} || \beta, \ell, J \rangle}{\sqrt{2J + 1}},$$

$$n = \frac{(2s + 1)}{(2\pi)^d} \int_{k < k_f} d^d k, \quad d \text{ dimensions,}$$

$$\{\Psi_s(\vec{x}), \Psi_{s'}^{\dagger}(\vec{y})\} = \delta^3(\vec{x} - \vec{y}) \delta_{ss'},$$

$$\Psi_s^{\dagger}(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} a_s^{\dagger}(\vec{k}), \quad \{\Psi_s(\vec{x}), a_{\alpha}^{\dagger}\} = \phi_{\alpha, s}(\vec{x}).$$

1. Consider a one-dimensional world where a type-**A** particle of mass **M** has zero momentum and decays to two type-**B** particles. The type-**B** particles are massless. The interaction is of the form

$$V = g \int dx (\Psi_B^\dagger(x) + \Psi_B(x))^2 [\Phi_A(x) + \Phi_A^\dagger(x)].$$

The field operators are defined by

$$\Phi_A^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} a_k^\dagger, \quad \Psi_B^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{E_k}} e^{-ikx} b_k^\dagger.$$

Calculate the rate at which the type-**A** particle decays into a type-**B** particle.

$$\begin{aligned} \langle f | H_{int} | i \rangle &= g \int dx \langle k, k' | \Psi_B^\dagger(x)^2 | 0 \rangle \cdot \langle 0 | \Phi_A(x) | k_A=0 \rangle \\ &= \frac{g \int dx e^{-i(k+k') \cdot x}}{L^{3/2} E_k} \cdot 2 = \frac{2g}{L^{1/2} E_k} \delta_{k, -k'} \\ &\quad \left(\begin{array}{l} \leftarrow 2 \text{ ways to connect } \Psi_B^2 \text{ to } k, k' \\ \delta(E_k + E_{k'} - Mc^2) \end{array} \right) \\ \Gamma_A &= \frac{2\pi}{\hbar} \frac{4g^2}{L E_k^2} \sum_{k, k'} \delta_{k, -k'} \\ &= \frac{2\pi}{\hbar} \frac{4g^2}{E_k^2 L} \frac{L}{2\pi} \int_0^\infty dk \delta(2E_k - Mc^2) \\ &\quad \delta(2\hbar ck - Mc^2) \\ &= \frac{2\pi}{\hbar} \frac{4g^2}{(\hbar ck)^2} \frac{1}{2\pi} \frac{1}{2\hbar c} \\ &= \frac{4g^2}{2\hbar^4 c^3 k^2} \quad \hbar ck = \frac{Mc^2}{2} \\ &= \frac{8g^2}{\hbar^2 M^2 c^5} \end{aligned}$$

(Extra work space for #1)

2. Consider the following matrix element,

$$\mathcal{M}(m_i, m_f) = \langle \ell_f = 2, m_f | P_x^2 + P_y^2 | \ell_i = 4, m_i \rangle.$$

- (a) For which combinations of m_i, m_f is \mathcal{M} non-zero?
 (b) If one were to calculate the matrix element

$$\mathcal{M}(m_i = 0, m_f = 0) = \langle \ell_f = 2, m_f = 0 | P_x^2 + P_y^2 | \ell_i = 4, m_i = 0 \rangle,$$

express the non-zero elements from (a) in terms of $\mathcal{M}(0, 0)$. You can leave the answer in terms of ratios of Clebsch-Gordan coefficients.

a)

m_i	m_f
2	2
1	1
0	0
-1	-1
-2	-2

b)

$$\mathcal{M}(m_i = m_f) = \frac{\binom{2\ell_f}{0 m_i; 2 m_f}}{\binom{2\ell_i}{0 0; 2 0}} \cdot \mathcal{M}(0, 0)$$

(Extra work space for #2)

3. Electrons are confined to a two-dimensional surface to move in the $x - y$ plane ($z = p_z = 0$). The areal density of electrons, number of electrons per area, is σ . Electrons of the same spin have the same energy until a magnetic field is added along the z axis. This gives the interaction,

$$\mathbf{H}_B = g_s \mu_B \mathbf{B} \frac{s_z}{\hbar},$$

where $\mu_B = e\hbar/2mc$, is the Bohr magneton and $g_s = 2$. Note the sign is positive above because the magnetic moment of the electron is negative (due to its negative charge).

- (a) In terms of m and σ , what are the Fermi energy, ϵ_{f0} , and the Fermi wave number, k_{f0} , when $\mathbf{B} = 0$?
- (b) What is the aerial magnetic moment density, M_z , for small fields?

$$M_z = \frac{1}{2} g_s \mu_B (\sigma_{\downarrow} - \sigma_{\uparrow}) = \chi B.$$

where σ_{\uparrow} is the aerial density of spin-up electrons. Express χ in terms of g_s, e, \hbar, m, c and k_f .

$$a) \quad \sigma = \frac{2}{4\pi} k_f^2, \quad k_f = \sqrt{2\pi\sigma}$$

$$\epsilon_f = \frac{\hbar^2 k_f^2}{2m}$$

$$b) \quad \Delta\sigma_{\uparrow} = \frac{2}{4\pi} k_f \frac{\Delta k_f}{2} = \frac{1}{4\pi} k_f \Delta k_f$$

$$\sigma_{\uparrow} - \sigma_{\downarrow} = \frac{1}{2\pi} k_f \Delta k_f$$

$$\Delta\epsilon_f = 2\mu_B B = \frac{\hbar^2 k_f \Delta k_f}{m}$$

$$\sigma_{\uparrow} - \sigma_{\downarrow} = \frac{1}{2\pi} k_f \frac{m}{\hbar^2 k_f} \cdot 2\mu_B B$$

$$= \frac{m}{\pi \hbar^2} \mu_B B$$

$$\chi = \frac{m \mu_B}{\pi \hbar^2} = \frac{e^2}{4\pi m c^2}$$

(Extra work space for #3)