FINAL EXAM(practice)

PHYSICS 851, FALL 2019

Monday/Wednesday, December 2/4, 9:10-10:00 AM

This exam is worth 0 points

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$$\int_{-\infty}^{\infty} dx\; e^{-x^2/2} = \sqrt{2\pi},$$

$$H = i\hbar\partial_t,\;\; \vec{P} = -i\hbar\nabla,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),$$

$$\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),$$

$$|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar},\;\; |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},$$

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(a^\dagger a + 1/2),$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,$$

$$\psi_0(x) = \frac{1}{(\pi b^2)^{1/4}}e^{-x^2/2b^2},\;\; b^2 = \frac{\hbar}{m\omega},$$

$$\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.$$

$$H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,$$

$$\text{For } V = \beta\delta(x-y),$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A},\;\; \vec{B} = \nabla\times\vec{A},$$

$$\omega_{\rm cyclotron} = \frac{eB}{mc},$$

$$e^{A+B} = e^A e^B e^{-C/2},\;\; \text{if } [A,B] = C,\; \text{and } [C,A] = [C,B] = 0,$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm 1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},$$

$$|N\rangle = |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n} \langle m|V|n\rangle + \cdots$$

$$E_N = \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \frac{|\langle m|V|n\rangle|^2}{\epsilon_m - \epsilon_n}$$

$$j_0(x) = \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x,$$

$$\frac{d}{dt}P_{i\to n}(t) = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i),$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}}\right|^2,$$

$$\sigma = \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k - \epsilon_r)^2 + (\hbar\Gamma_R/2)^2},$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm single}\tilde{S}(\vec{q}),\quad \tilde{S}(\vec{q}) = \left|\sum_{\delta\vec{a}} e^{i\vec{q}\cdot\delta\vec{a}}\right|^2,$$

$$e^{i\vec{k}\cdot\vec{r}} = \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$$

$$P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi),$$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/3,$$

$$f(\Omega) \equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta)$$

$$\psi_{\vec{k}}(\vec{r})|_{R\to\infty} = e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega),$$

$$\frac{d\sigma}{d\Omega} = |f(\Omega)|^2,$$

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi},$$

$$L_\pm|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m\pm1)}|\ell,m\pm1\rangle.$$

5. A neutron and proton occupy the ground state of a harmonic oscillator. The particles then feel two additional sources of interaction. First, they have a spin-spin interaction,

$$\boldsymbol{V_{\mathrm{s.s.}} = \alpha \mathbf{S_n} \cdot \mathbf{S_p}},$$

and secondly, they experience an external magnetic field

$$\boldsymbol{V_b = -\mathbf{B} \cdot (\mu_n \mathbf{S_n} + \mu_p \mathbf{S_p})}\,.$$

(a) (5 pts) If the magnetic field is zero, what are the energy levels? Note the degeneracy of each level.

(b) (5 pts) If the magnetic field is non-zero but the spin-spin coupling is neglected ($\boldsymbol{\alpha = 0}$), what are the energy eigenvalues? Again, note the degeneracy of each level.

(c) (10 pts) When $\boldsymbol{\alpha \neq 0}$, $\boldsymbol{B \neq 0}$, and $\vec{\boldsymbol{B}}$ points along the $\boldsymbol{z}$ axis, which of the following operators commute with the Hamiltonian. Circle the correct choices, and no credit is given for wrong answers with good reasoning. (Note: $\vec{\boldsymbol{J}} \equiv \vec{\boldsymbol{S}}_{\mathbf{n}} + \vec{\boldsymbol{S}}_{\mathbf{p}}$)

i. $|\vec{\boldsymbol{J}}|^2 = \boldsymbol{J_x^2 + J_y^2 + J_z^2}$.

ii. $\boldsymbol{J_z}$

iii. $\boldsymbol{J_x}$

iv. $\boldsymbol{S_{\mathbf{n},z}}$

v. $\boldsymbol{S_{\mathbf{n},x}}$

vi. $|\vec{\boldsymbol{S}}_{\mathbf{n}}|^2$

vii. $\vec{\boldsymbol{S}}_{\mathbf{n}} \cdot \vec{\boldsymbol{S}}_{\mathbf{p}}$

6. A particle of mass m scatters off a target with a spherically symmetric potential,

$$V(r) = \beta\delta(r - R).$$

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the momentum p.

(b) (5 pts) What is the cross-section in the limit that $p \to 0$?

(Extra work space for #2)

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7. A two-level system is initially in the ground state. The initial Hamiltonian is

$$H_0 = V_0 \sigma_z.$$

An interaction is added,

$$V(t) = \beta(t)\sigma_x, \quad \beta(t < 0) = 0, \; \beta(t \to \infty) = \beta_0.$$

(a) (5 pts) What is the ground state wave function for $t < 0$?

(b) (5 pts) What is the ground state wave function for $t \to \infty$?

(c) (5 pts) If the interaction is turned on suddenly, what is the probability the system is in the new ground state as $t \to \infty$?

(d) (5 pts) If the interaction is turned on slowly, what is the probability the system is in the new ground state as $t \to \infty$?

(e) (10 pts) To first order in perturbation theory, what is the new ground state wave function?

8. (30 pts) Consider a *Brian* particle of mass m confined to a one-dimensional potential,

$$V(x) = \begin{cases} \infty, & x < -a \\ 0, & -a < x < a \\ \infty, & x > a \end{cases} .$$

It can decay to a *Brianna* particle of the same mass, but the Brianna particle does not feel the potential. The Hamiltonian matrix element responsible for the decay is

$$\langle 0, \text{Brian}|V|k, \text{Brianna}\rangle = \frac{\alpha e^{-k^2 b^2/2}}{\sqrt{L}},$$

where the momentum of the Brianna particle is $\hbar k$, the large length of the plane wave $|k\rangle$ is L, and the constant α is small. What is the Brian-particle decay rate? Present your answer in terms of α, a, b, V and m.

9. (20 pts) Consider a particle of mass $\boldsymbol{m}$ in a one-dimensional harmonic oscillator potential with fundamental frequency $\boldsymbol{\omega}$,

$$\boldsymbol{H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2.}$$

To second order in perturbation theory, what is the correction to the ground state energy when the perturbation

$$\boldsymbol{V = \beta P,}$$

is added to the system.

(Extra work space for #5)

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10. In a two-level system, a system finds itself in an eigenstate of $\boldsymbol{\sigma_y}$ with eigenvalue $\mathbf{+1}$

    (a) (10 pts) Write the density matrix $\boldsymbol{\rho_+}$.

    (b) (5 pts) What is $\boldsymbol{\rho_+^2}$.

    (c) (5 pts) If one is now incoherently occupying eigenstates with both eigenvalues of $\boldsymbol{\sigma_y}$ with equal probability, what is the new density matrix?

    (d) (5 pts) What is the square of this density matrix?

(Extra work space for #6)

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11. A particle of mass $\boldsymbol{m}$ and charge $\boldsymbol{e}$ is placed in a region with uniform magnetic field $\boldsymbol{B}$ along the $\boldsymbol{z}$ axis.

   (a) (5 pts) Write the vector potential that describes the magnetic field such that $\vec{\boldsymbol{A}}$ is in the $\hat{\boldsymbol{y}}$ direction.

   (b) (5 pts) Write the Hamiltonian with this vector potential.

   (c) (5 pts) Circle the quantities that commute with the Hamiltonian?

      i. $\boldsymbol{P_x}$
      ii. $\boldsymbol{P_y}$
      iii. $\boldsymbol{P_z}$
      iv. $\boldsymbol{P_x - eA_x/c}$
      v. $\boldsymbol{P_y - eA_y/c}$
      vi. $\boldsymbol{P_z - eA_z/c}$

   (d) (10 pts) Consider the notation where the eigenstate wave functions for a one-dimensional Harmonic oscillator Hamiltonian,

$$\boldsymbol{H = -\frac{\hbar^2 \partial_u^2}{2m} + \frac{1}{2} m\omega^2 u^2,}$$

   are labeled $\boldsymbol{\phi_n(m, \omega, u)}$. Write the **most general** three-dimensional wavefunctions that are eigenstates of the Hamiltonian with the vector potential $\vec{\boldsymbol{A}}$ used above. This form should incorporate ALL possible eigenstates. Express your answer in terms of $\boldsymbol{\phi_n}$ plane wave forms. Be sure to list all the quantum numbers that are used to span the space. Also, in terms of these quantum numbers, express the eigen-energies of the wave functions.