FINAL EXAM(practice)

PHYSICS 851, FALL 2019

Thursday, December 12, 7:45-9:45 AM

This exam is worth 0 points

$$\boldsymbol{H = i\hbar\partial_t,}$$

$$\boldsymbol{\vec{P} = -i\hbar\nabla,}$$

$$\boldsymbol{\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},}$$

$$\boldsymbol{U(t,-\infty) = 1 + \frac{-i}{\hbar}\int_{-\infty}^{t} dt'\; V(t')U(t',-\infty),}$$

$$\boldsymbol{\langle x|x'\rangle = \delta(x-x'),\; \langle p|p'\rangle = \frac{1}{2\pi\hbar}\delta(p-p'),}$$

$$\boldsymbol{|p\rangle = \int dx\; |x\rangle e^{ipx/\hbar}, \quad |x\rangle = \int \frac{dp}{2\pi\hbar}|p\rangle e^{-ipx/\hbar},}$$

$$\boldsymbol{H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega(a^\dagger a + 1/2),}$$

$$\boldsymbol{a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P,}$$

$$\boldsymbol{\rho(\vec{r},t) = \psi^*(\vec{r}_1,t_1)\psi(\vec{r}_2,t_2)}$$

$$\boldsymbol{\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t))}$$

$$\boldsymbol{- \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2.}$$

$$\boldsymbol{H = \frac{(\vec{P} - e\vec{A}/c)^2}{2m} + e\Phi,}$$

$$\textbf{For } \boldsymbol{V = \beta\delta(x-y),}$$

$$\boldsymbol{-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\psi(x)|_{\cancel{x}\,y+\epsilon} - \frac{\partial}{\partial x}\psi(x)|_{y-\epsilon}\right) = -\beta\psi(y),}$$

$$\boldsymbol{\vec{E} = -\nabla\Phi - \frac{1}{c}\partial t\vec{A}, \quad \vec{B} = \nabla\times\vec{A},}$$

$$\boldsymbol{\omega_{\text{cyclotron}} = \frac{eB}{mc},}$$

$$\boldsymbol{e^{A+B} = e^A e^B e^{-C/2}, \;\; \text{if } [A,B] = C, \text{ and } [C,A] = [C,B] = 0,}$$

$$\boldsymbol{Y_{0,0} = \frac{1}{\sqrt{4\pi}}}$$

$$\boldsymbol{Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta}$$

$$\boldsymbol{Y_{1,\pm1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\pm\phi},}$$

$$
\begin{aligned}
|N\rangle &= |n\rangle - \sum_{m\neq n} |m\rangle \frac{1}{\epsilon_m - \epsilon_n}\langle m|V|n\rangle + \cdots \\
E_N &= \epsilon_n + \langle n|V|n\rangle - \sum_{m\neq n} \cancel{|m\rangle} \frac{1}{\epsilon_m - \epsilon_n}\Big|\langle m|V|n\rangle\Big|^2 \\
j_0(x) &= \frac{\sin x}{x},\ n_0(x) = -\frac{\cos x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x},\ n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \\
j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x,\ n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x, \\
\frac{d}{dt}P_{i\to n}(t) &= \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i), \\
\frac{d\sigma}{d\Omega} &= \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\,\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2, \\
\sigma &= \frac{(2S_R+1)}{(2S_1+1)(2S_2+1)}\frac{4\pi}{k^2}\frac{(\hbar\Gamma_R/2)^2}{(\epsilon_k-\epsilon_r)^2+(\hbar\Gamma_R/2)^2}, \\
e^{i\vec{k}\cdot\vec{r}} &= \sum_\ell (2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta), \\
P_\ell(\cos\theta) &= \sqrt{\frac{4\pi}{2\ell+1}}Y_{\ell,m=0}(\theta,\phi), \\
P_0(x) = 1,\ P_1(x) &= x,\ P_2(x) = (3x^2-1)/3, \\
f(\Omega) &\equiv \sum_\ell (2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \frac{1}{k}P_\ell(\cos\theta) \\
\psi_{\vec{k}}(\vec{r})|_{R\to\infty} &= e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r}f(\Omega), \\
\frac{d\sigma}{d\Omega} &= |f(\Omega)|^2, \\
\sigma &= \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell, \\
\int_{-\infty}^{\infty} dx\ e^{-x^2/2} &= \sqrt{2\pi}, \\
L_\pm|\ell,m\rangle &= \sqrt{\ell(\ell+1)-m(m\pm1)}|\ell,m\pm1\rangle.
\end{aligned}
$$

1. A neutron and proton occupy the ground state of a harmonic oscillator. The particles then feel two additional sources of interaction. First, they have a spin-spin interaction,

$$\boldsymbol{V_{\mathrm{s.s.}} = \alpha \mathbf{S_n} \cdot \mathbf{S_p},}$$

and secondly, they experience an external magnetic field

$$\boldsymbol{V_b = -\mathbf{B} \cdot (\mu_n \mathbf{S_n} + \mu_p \mathbf{S_p})\,.}$$

(a) (5 pts) If the magnetic field is zero, what are the energy levels? Note the degeneracy of each level.

(b) (5 pts) If the magnetic field is non-zero but the spin-spin coupling is neglected ($\boldsymbol{\alpha = 0}$), what are the energy eigenvalues? Again, note the degeneracy of each level.

(c) (10 pts) When $\boldsymbol{\alpha \neq 0}$, $\boldsymbol{B \neq 0}$, and $\vec{\boldsymbol{B}}$ points along the $\boldsymbol{z}$ axis, which of the following operators commute with the Hamiltonian. Circle the correct choices, and no credit is given for wrong answers with good reasoning. (Note: $\vec{\boldsymbol{J}} \equiv \vec{\boldsymbol{S}}_{\mathbf{n}} + \vec{\boldsymbol{S}}_{\mathbf{p}}$)

i. $|\vec{\boldsymbol{J}}|^2 = \boldsymbol{J_x^2 + J_y^2 + J_z^2}$.

ii. $\boldsymbol{J_z}$

iii. $\boldsymbol{J_x}$

iv. $\boldsymbol{S}_{\mathbf{n},\boldsymbol{z}}$

v. $\boldsymbol{S}_{\mathbf{n},\boldsymbol{x}}$

vi. $|\vec{\boldsymbol{S}}_{\mathbf{n}}|^2$

vii. $\vec{\boldsymbol{S}}_{\mathbf{n}} \cdot \vec{\boldsymbol{S}}_{\mathbf{p}}$

a) only feel s.s. interaction

$$V = \alpha \vec{S}_n \cdot \vec{S}_p$$

$$= \frac{\hbar^2}{2}\alpha \left(S(S+1) - s(s+1) - s(s+1) \right)$$

$$E = \begin{cases} \frac{\hbar^2}{4}\alpha, & S = 1 \quad (\text{degen} = 3) \\ -\frac{3\hbar^2}{4}\alpha, & S = 0 \quad (\text{degen} = 1) \end{cases}$$

b) only B

$$V = -\mu_n \vec{S}_n \cdot B - \mu_p \vec{S}_p \cdot \vec{B}$$

$$= -\mu_n \hbar B m_n - \mu_p \hbar B m_p$$

$$E = \frac{\hbar B}{2} \begin{cases} -\mu_n - \mu_p, & m_p = m_n = \frac{1}{2} \\ \mu_p + \mu_n, & m_p = m_n = -\frac{1}{2} \\ \mu_n - \mu_p, & m_p = \frac{1}{2}, \ m_n = -\frac{1}{2} \\ \mu_p - \mu_n, & m_n = -\frac{1}{2}, \ m_n = \frac{1}{2} \end{cases}$$

c) J_z $\{ \ |\vec{S}_n|^2$

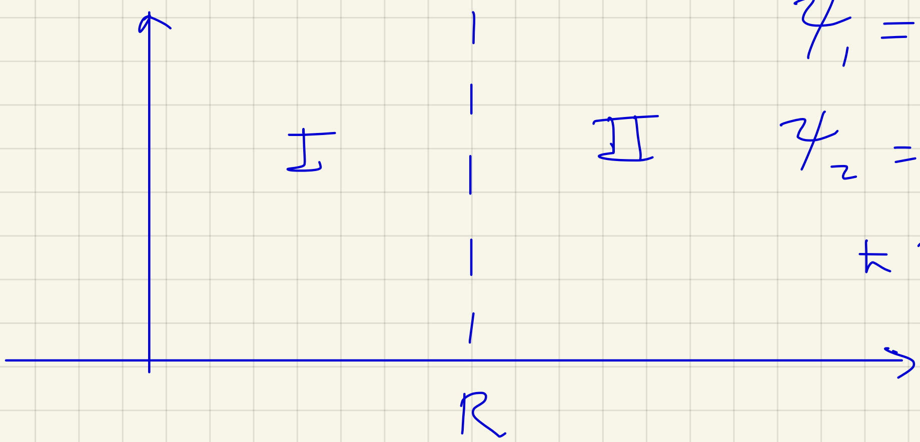
2. A particle of mass m scatters off a target with a spherically symmetric potential,

$$V(r) = \beta\delta(r - R).$$

(a) (10 pts) Find the $\ell = 0$ phase shift as a function of the momentum p.

(b) (5 pts) What is the cross-section in the limit that $p \to 0$?

$$V = \beta\,\delta(R - r)$$



$$\psi_1 = A \sin kr$$

$$\psi_2 = \sin(kr + \delta)$$

$$\hbar k = p$$

ⓐ

$$A \sin(kR) = \sin(kR + \delta)$$

$$k \cdot A \cos(kR) = k \cos(kR + \delta) - \frac{2m\beta}{\hbar^2} A \sin kR$$

$$\frac{\sin kR}{k \cos kR + \frac{2m\beta}{\hbar^2} \sin kR} = \frac{1}{k} \tan(kR + \delta)$$

$$\delta = -kR + \tan^{-1}\left\{ \frac{\sin kR}{\cos kR + \frac{2m\beta}{\hbar^2 k} \sin kR} \right\}$$

$$k = p/\hbar$$

ⓑ As $k \to 0$,

$$\delta = -kR + \left\{ \frac{kR}{1 + \frac{2m\beta R}{\hbar^2}} \right\}$$

$$\sigma = 4\pi R^2 \left\{ \frac{1}{1 + \frac{2m\beta}{\hbar^2}} - 1 \right\}^2$$

$$= 4\pi R^2 \frac{(2m\beta/\hbar^2)^2}{\left(1 + \frac{2m\beta}{\hbar^2}\right)^2}$$

3. A two-level system is initially in the ground state. The initial Hamiltonian is

$$\boldsymbol{H_0 = V_0 \sigma_z.}$$

An interaction is added,

$$\boldsymbol{V(t) = \beta(t)\sigma_x, \quad \beta(t < 0) = 0, \ \beta(t \to \infty) = \beta_0.}$$

(a) (5 pts) What is the ground state wave function for $\boldsymbol{t < 0}$?

(b) (~~5~~ 10 pts) What is the ground state wave function for $\boldsymbol{t \to \infty}$?

(c) (5 pts) If the interaction is turned on suddenly, what is the probability the system is in the new ground state as $\boldsymbol{t \to \infty}$?

(d) (5 pts) If the interaction is turned on slowly, what is the probability the system is in the new ground state as $\boldsymbol{t \to \infty}$?

(e) (10 pts) To first order in perturbation theory, what is the new ground state wave function?

$$V(t) = \beta(t)\,\sigma_x \;, \quad H_0 = V_0\,\sigma_z$$

(a) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $E = -V_0$

(b) $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $H = V_0\sigma_z + \beta_0\sigma_x$

$$E_0 = -\sqrt{V_0^2 + \beta_0^2}$$

$$-\sqrt{V_0^2+\beta_0^2}\begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} V_0 + \beta_0 x \\ -V_0 x + \beta_0 \end{pmatrix}$$

$$-\sqrt{V_0^2+\beta_0^2} = V_0 + \beta_0 x$$

or

$$-\sqrt{V_0^2+\beta_0^2}\,x = -V_0 x + \beta_0$$

$$x = -\frac{V_0 + \sqrt{V_0^2+\beta_0^2}}{\beta_0}$$

normalising

$$\psi_0 = \frac{1/\beta_0}{\sqrt{1 + \frac{\left(V_0 + \sqrt{V_0^2+\beta_0^2}\right)^2}{\beta_0^2}}}\begin{pmatrix} \beta_0 \\ -\left(V_0 + \sqrt{V_0^2+\beta_0^2}\right) \end{pmatrix}$$

$$= \frac{1}{\left(2\beta_0^2 + 2V_0^2 + 2V_0\sqrt{V_0^2+\beta_0^2}\right)^{1/2}}\begin{pmatrix} -\beta_0 \\ V_0 + \sqrt{V_0^2+\beta_0^2} \end{pmatrix}$$

(c)

$$\frac{\left(V_0 + \sqrt{\beta_0^2 + V_0^2}\right)^2}{\left(2\beta_0^2 + 2V_0^2 + 2V_0\sqrt{\beta_0^2+V_0^2}\right)}$$

$$= \frac{2V_0^2 + \beta_0^2 + 2V_0\sqrt{\beta_0^2+V_0^2}}{2V_0^2 + 2\beta_0^2 + 2V\sqrt{\beta_0^2+V_0^2}}$$

(d) 100%

(e)

$$|\psi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\beta_0}{2V_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. (30 pts) Consider a *Brian* particle of mass $\boldsymbol{m}$ confined to a one-dimensional potential,

$$\boldsymbol{V(x)} = \begin{cases} \infty, & \boldsymbol{x < -a} \\ 0, & \boldsymbol{-a < x < a} \\ \infty, & \boldsymbol{x > a} \end{cases} .$$

It can decay to a *Brianna* particle of the same mass, but the Brianna particle does not feel the potential. The Hamiltonian matrix element responsible for the decay is

$$\langle \mathbf{0, Brian}|\boldsymbol{V}|\boldsymbol{k}, \mathbf{Brianna}\rangle = \frac{\boldsymbol{\alpha e^{-k^2b^2/2}}}{\sqrt{\boldsymbol{L}}},$$

where the momentum of the Brianna particle is $\hbar\boldsymbol{k}$, the large length of the plane wave $|\boldsymbol{k}\rangle$ is $\boldsymbol{L}$, and the constant $\boldsymbol{\alpha}$ is small. What is the Brian-particle decay rate? Present your answer in terms of $\boldsymbol{\alpha}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{V}$ and $\boldsymbol{m}$.

$$R = \sum_k \frac{2\pi}{\hbar} |\langle \text{Brian} | V | \text{Brianna} \rangle|^2 \delta(\mathcal{E}_{\text{Brian}} - \mathcal{E}_{\text{Brianna}})$$

$$= \frac{2\pi}{\hbar} \alpha^2 \frac{e^{-k^2 b^2}}{L} L \int_{-\infty}^{\infty} \frac{dk}{2\pi} \delta\left(\mathcal{E} - \frac{\hbar^2}{2m} k^2\right)$$

$$= \frac{1}{\hbar} \alpha^2 e^{-k^2 b^2} \frac{m}{\hbar^2 k}$$

$$= \frac{m}{\hbar^3 k} |\alpha|^2 e^{-k^2 b^2}$$

$$\frac{\hbar^2 k^2}{2m} = \mathcal{E}$$

$$k = \sqrt{\frac{2m\mathcal{E}}{\hbar^2}}$$

$$= \frac{m^{1/2} |\alpha|^2}{\hbar^2 (2E)^{1/2}} e^{-k^2 b^2}$$

5. (20 pts) Consider a particle of mass $\boldsymbol{m}$ in a one-dimensional harmonic oscillator potential with fundamental frequency $\boldsymbol{\omega}$,

$$\boldsymbol{H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2.}$$

To second order in perturbation theory, what is the correction to the ground state energy when the perturbation

$$\boldsymbol{V = \beta P,}$$

is added to the system.

$$p = \frac{i(a^{+} - a)}{\sqrt{2}} \sqrt{\hbar m \omega}$$

$$\Delta E = -\frac{|\langle 1 | p | 0 \rangle|^2}{\hbar \omega}$$

$$= -\frac{\beta^2}{2} \frac{\hbar m \omega}{\hbar \omega} = -\frac{\beta^2 m}{2}$$

6. In a two-level system, a system finds itself in an eigenstate of $\boldsymbol{\sigma_y}$ with eigenvalue $\mathbf{+1}$

   (a) (10 pts) Write the density matrix $\boldsymbol{\rho_+}$.

   (b) (5 pts) What is $\boldsymbol{\rho_+^2}$.

   (c) (5 pts) If one is now incoherently occupying eigenstates with both eigenvalues of $\boldsymbol{\sigma_y}$ with equal probability, what is the new density matrix?

   (d) (5 pts) What is the square of this density matrix?

a $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\rho_+ = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

b $\rho_+^2 = \rho_+ = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

c $$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

d $\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

7. A particle of mass m and charge e is placed in a region with uniform magnetic field B along the z axis.

   (a) (5 pts) Write the vector potential that describes the mag. field ~~potential~~ such that $\vec{A}$ is in the $\hat{y}$ direction.

   (b) (5 pts) Write the Hamiltonian with this vector potential.

   (c) (5 pts) Circle the quantities that commute with the Hamiltonian?

      i. P_x

      ii. P_y

      iii. P_z

      iv. $P_x - eA_x/c$

      v. $P_y - eA_y/c$

      vi. $P_z - eA_z/c$

   (d) (10 pts) Consider the notation where the eigenstate wave functions for a one-dimensional Harmonic oscillator Hamiltonian,

$$H = -\frac{\hbar^2 \partial_u^2}{2m} + \frac{1}{2} m\omega^2 u^2,$$

   are labeled $\phi_n(m, \omega, u)$. Write the **most general** three-dimensional wavefunctions that are eigenstates of the Hamiltonian with the vector potential $\vec{A}$ used above. This form should incorporate ALL possible eigenstates. Express your answer in terms of ϕ_n plane wave forms. Be sure to list all the quantum numbers that are used to span the space. Also, in terms of these quantum numbers, express the eigen-energies of the wave functions.

(a) $\vec{A} = B x \hat{y}$

(b) $$\mathcal{H} = \frac{p_x^2}{2m} + \frac{(p_y - eBx/c)^2}{2m} + \frac{p_z^2}{2m}$$

(c) p_y, p_z, $p_z - \frac{eA_z}{c}$

(d) $$H = \frac{p_x^2}{2m} + \frac{e^2B^2}{2mc^2}\left(x - \left(p_y c/eB\right)\right)^2 + \frac{p_z^2}{2m}$$

$$\psi(n, k_y, k_z) = e^{ik_z z} e^{ik_y y} \phi_n\left(m, \omega = \frac{eB}{mc}, x - \frac{p_y c}{eB}\right)$$

$$\mathcal{E}_{n,k_y,k_z} = \frac{\hbar^2 k_z^2}{2m} + \left(n + \frac{1}{2}\right)\hbar \frac{eB}{mc}$$