SUBJECT EXAM

PHYSICS 851/852, SPRING 2001

1. (10 points) Consider a two-component system described by the spinor,

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

The system evolves under the influence of a Hamiltonian,

$$H = H_0 + \hbar\omega\sigma_x.$$

If the system begins life in the state

$$\psi(t=0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},$$

find $P_\uparrow(t)$, the probability of being in the state

$$\psi_\uparrow \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2. A particle of mass m moves under the influence of a repulsive spherically symmetric potential,

$$V(r) = \begin{cases} V_0, & r < a \\ 0, & r > a \end{cases}$$

(a) (10 points) Find the s-wave phase shift $\delta(E)$ for energies $E < V_0$.

(b) (5 points) What is the cross section for scattering in the limit $E \to 0$.

3. (10 points) Express the state $|s = 1/2, \ell = 2, m_s = 1/2, m_\ell = 1\rangle$ as a linear combination of eigenstates of total angular momentum J and projection M.

4. (15 points) Two types (*ted* and *alice*) of non-relativistic spin-1/2 fermions have equal mass m and move in a **TWO-DIMENSIONAL** world. They can undergo a reaction $ted + \gamma \leftrightarrow alice + \gamma'$, where γ refers to a photon. A macroscopic number are placed in a large box of area A that conserves the net number $N = N_{ted} + N_{alice}$, but allows photons to escape. The particles feel different potentials within the box,

$$\begin{aligned} V_{ted}(x,y) &= V_{ted} \\ V_{alice}(x,y) &= 0. \end{aligned}$$

After equilibrating at zero temperature, find N_{ted} and N_{alice} in terms of N, A, m and V_{ted}. (Assume $V_{\rm ted}$ is much less than the Fermi energy.)

5. A *bob* particle of mass m is in the first excited state of a **ONE-DIMENSIONAL** harmonic oscillator characterized by frequency ω. It can decay to the ground state via the emission of a *carol* particle which is massless and spinless. The potential responsible for the decay is

$$V = g \int dx \; \Psi^\dagger(x)\Phi(x)\Psi(x),$$

where Ψ and Φ are field operators for *bob* and *carol* particles respectively,

$$\begin{aligned} \Psi(x) &= \frac{1}{\sqrt{L}} \sum_k b_k e^{-ikx} = \sum_n \phi_n(x) b_n \\ \Phi(x) &= \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{kc}} \left(c_k^\dagger e^{ikx} + c_k e^{-ikx} \right), \end{aligned}$$

where $b_k^\dagger$ and $c_k^\dagger$ create *bob*s and *carol*s with momentum $\hbar k$, and $b_n^\dagger$ would create *bob*s into any state n which is part of an orthonormal basis described by wave functions $\phi_n(x)$.

(a) (5 points) What is the dimension of g?

(b) (10 points) Calculate $\langle k, 0|V|1\rangle$, the matrix element for decay of a *bob* from the first excited state into the ground state via emission of a *carol* with momentum k.

(c) (10 points) In terms of $\hbar$, m, ω and $\mathcal{M} \equiv \sqrt{L}\langle k, 0|V|1\rangle$, calculate the lifetime of the first excited state.

Potentially useful information:

$$\psi_0(x) = \frac{1}{\pi^{1/4} a^{1/2}} e^{-x^2/(2a^2)}, \quad a^2 = \frac{\hbar}{m\omega} \tag{1}$$

$$\psi_1(x) = x \frac{\sqrt{2}}{a} \psi_0(x) \tag{2}$$

$$E = \hbar k c, \text{ for a massless particle.} \tag{3}$$

6. (15 points) Consider eigenstates of the hydrogen atom whose angular wave functions are described by ℓ and m_ℓ. All other information about the eigenstate (e.g. spin and radial wave functions) are referred to by α and β. For each of the matrix elements below,

   (a) $\langle \alpha, \ell = 2, m_\ell = 0 |\ r^2\ | \beta, \ell = 0, m_\ell = 0 \rangle$
   (b) $\langle \alpha, \ell = 4, m_\ell = 0 |\ (x + iy)^2\ | \beta, \ell = 2, m_\ell = 0 \rangle$
   (c) $\langle \alpha, \ell = 2, m_\ell = 2 |\ z^2\ | \beta, \ell = 0, m_\ell = 0 \rangle$
   (d) $\langle \alpha, \ell = 3, m_\ell = 3 |\ z^2\ | \beta, \ell = 3, m_\ell = 3 \rangle$
   (e) $\langle \alpha, \ell = 3, m_\ell = 2 |\ x\ | \beta, \ell = 3, m_\ell = 1 \rangle$,

   choose one of the following statements.

   A. Might be non-zero.
   B. Must be zero due to parity.
   C. Must be zero due to time-reversal.
   D. Must be zero due angular momentum conservation, a.k.a. the Wigner Eckart theorem.
   E. Must be zero due to conservation of electric charge.

7. Consider the quantum state

$$|\eta\rangle \equiv e^{-\eta^*\eta/2} e^{\eta a^\dagger} |0\rangle.$$

   (a) (5 points) Calculate $\langle 0|a|\eta\rangle$.
   (b) (5 points) Calculate $\langle \eta|(a^\dagger)^3 a^2|\eta\rangle$

   (You can use the fact that $\langle \eta|\eta\rangle = 1$.