1.11 Problems

1. Photons, traveling along the z axis can be polarized either linearly along the x or y axis, or a linear combination of the two states. Write the operator that rotates the states by 45° about the z axis in terms of $|x\rangle$, $|y\rangle$ and the corresponding kets.

1. $|45^\circ\rangle = R|x\rangle$

$|135^\circ\rangle = R|y\rangle$

$$R = |45^\circ\rangle\langle x| + |135^\circ\rangle\langle y|$$

2. Choosing the basis,

$$|x\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

write the matrix that rotates the states by ϕ about the z axis.

$$|\varphi\rangle = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = R(\varphi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = R\left(\varphi - \frac{\pi}{2}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{pmatrix}$$

3. Right-hand circularly polarized (RCP) light is made of a linear combination of x and y polarized light.

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right).$$

Light traveling along the z axis passes through a thin slab of thickness t whose index of refraction, $k = n\omega/c$, is different for light polarized in the x and y directions. In terms of n_x, n_y and t find the polarization vector for light which enters the slab as right-circularly polarized.
HINT: The wave has a form $e^{-i\omega t+ikz}$. The two components have the same ω but different k while in the medium.

Let $\omega = k/n$, $k = n\omega$,

$$\psi = \frac{e^{i\omega\tau}}{\sqrt{2}}\begin{pmatrix} e^{-ik_x z} \\ i\,e^{-ik_y z} \end{pmatrix}$$

$$\psi(t) = \frac{e^{i\omega\tau}}{\sqrt{2}}\, e^{-ik_x t}\begin{pmatrix} 1 \\ i\,e^{-i(n_y-n_x)\omega t} \end{pmatrix}$$

4. Find the density matrix for right-circularly polarized light in the basis defined above.

$$|\psi\rangle\langle\psi| = \begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

5. Using the basis described above, write the density matrix for light that is an incoherent mixture, 50% polarized along the x direction and 50% along the y direction.

$$|\rho\rangle = 0.5\, |R\rangle\langle R| + 0.5\cdot |L\rangle\langle L|$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

6. Considering a photon's polarization, calculate $\langle x|\mathcal{R}(\theta)|x\rangle$ for $\theta=\pi/2, \pi, 2\pi$, where the rotation is about the z axis.

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$R(\theta)|x\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \langle x|R(\theta)|x\rangle = \cos\theta$$

$$\langle x|R\left(\tfrac{\pi}{2}\right)|x\rangle = \emptyset, \quad \langle x|R(\pi)|x\rangle = -1$$

$$\langle x|R(2\pi)|x\rangle = 1$$

7. For a spin 1/2 particle, calculate $\langle z,+|\mathcal{R}(\theta)|z,+\rangle$, for the same angles when the rotation is about the y axis.

$$|z,+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$R(\theta) = \cos\theta/2 - i\sin\frac{\theta}{2}\,\sigma_y$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R(\theta)|z,+\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$\langle z,+|R(\theta)|z,+\rangle = \cos(\theta/2)$$

$$\langle z,+|R(\pi/2)|z,+\rangle = 1/\sqrt{2}$$

$$\langle z,+|R(\pi)|z,+\rangle = 0$$

$$\langle z,+|R(2\pi)|z,+\rangle = -1$$

8. Show that the unit matrix $\mathbb{I}$, which can be considered as an operator, is unchanged by a unitary transformation. Begin with the fact that for any matrix $\mathcal{M}$, $\mathcal{M}\mathbb{I} = \mathbb{I}\mathcal{M} = \mathcal{M}$.

$$I' = U I U^{+} = U U^{+} I = I$$

unitary transformation. Begin with the fact that for any matrix $\mathcal{M}$, $\mathcal{M}\mathbb{I} = \mathbb{I}\mathcal{M} = \mathcal{M}$.

9. Consider the rotation matrix for rotating Pauli spinors by an angle $90°$ about the z axis. Using Eq. (1.44),

$$U = e^{-i\sigma_z\pi/4} = \frac{1}{\sqrt{2}}(1 - i\sigma_z).$$

(a) Using the commutator and anti-commutator relations for the $\boldsymbol{\sigma}$ matrices, show that the transformation of σ_x is

$$U\sigma_x U^{\dagger} = \sigma_y.$$

(b) Show that rotating the state, $|+, x\rangle$, which refers to an eigenstate of σ_x with eigenvalue of +1, gives

$$U|+, x\rangle = |+, y\rangle,$$

which is the eigenstate of σ_y with eigenvalue +1.

(a) $U \sigma_x U^{+} = \frac{1}{2}(1 - i\sigma_z)\sigma_x(1 + i\sigma_z)$

$= \frac{1}{2}(\sigma_x + \sigma_y)(1 + i\sigma_z)$

$= \frac{1}{2}\sigma_x + \frac{1}{2}\sigma_y + \frac{1}{2}\sigma_y - \frac{1}{2}\sigma_x = \sigma_y$

(b) $U|x+\rangle = \frac{1}{\sqrt{2}}(|x,+\rangle - i\sigma_z|x,+\rangle)$

$\sigma_y(U|x+\rangle) = \frac{1}{\sqrt{2}}\sigma_y|x,+\rangle + \frac{1}{\sqrt{2}}\sigma_x|x,+\rangle$

$= \frac{1}{\sqrt{2}}|x,+\rangle + \frac{1}{\sqrt{2}}\sigma_y\sigma_x|x,+\rangle$

$= \frac{1}{\sqrt{2}}|x,+\rangle - \frac{i}{\sqrt{2}}\sigma_y|x,+\rangle$

$= U|x,+\rangle$

thus $U|x,+\rangle$ is eigenstate of σ_y

& has eigen value $= 1$, $\therefore = |y,+\rangle$

which is the eigenstate of σ_y with eigenvalue +1.

10. Consider some Hermitian $N \times N$ matrix K_{ij}, with eigenvalues $\lambda^{(n)}$ and the corresponding normalized eigenvectors $v^{(n)}$,

$$Kv^{(n)} = \lambda^{(n)} v^{(n)}.$$

The N eigenvectors each have N components, $v_i^{(n)}$. Create an $N \times N$ matrix

$$U_{ij} = v_j^{(i)}.$$

Thus, one is making a matrix by having each row be one of the eigenvectors.

(a) Show that U is unitary.

$$U^{+}_{ji} = v_j^{(i)}$$

$$U_{ij} U^{+}_{jk} = v_j^{(i)*} v_j^{(k)} = \delta_{ik} \quad \checkmark$$

(b) Show that the j^{th} component of the vector $Uv^{(n)}$ is

$$(Uv^{(n)})_j = \delta_{nj},$$

Thus, the vectors $Uv^{(n)}$ are

$$(Uv^{(1)}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (Uv^{(2)}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots$$

Now, consider the matrix

$$K' = (UKU^\dagger),$$

and have it act on the vectors above. Show that

$$K'(Uv^{(n)}) = \lambda_n (Uv^{(n)}).$$

This shows that the vectors $(Uv^{(n)})$ are eigenvectors of the matrix K' with eigenvalues λ_n. Given that the eigenvectors are of the simple form above, the matrix $(UKU^\dagger)$ must be diagonal. Thus the matrix U defined above provides the unitary matrix for transforming the matrix K into its diagonal form.

b) $$(Uv^{(n)})_j = U_{ji} v_i = v_i^{(j)} v_i^{(n)} = \underline{\delta_{jn}} \quad \checkmark$$

$$K' = UKU^{+}$$

$$K' U v^{(n)} = U U^{+} K' U v^{(n)} = U K v^{(n)} = \lambda^{(n)} U v^{(n)}$$

11. Consider the matrix:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) What are the eigenvalues of $\mathcal{M}$?

(b) What are the eigenvectors of $\mathcal{M}$?

$$M = \begin{pmatrix} 1 & 0\,0 \\ 0 & \begin{pmatrix} \sigma_x \end{pmatrix} \\ 0 & \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \text{ in rotation}$$

eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

eigenvalues are

$$1, \quad 1, \quad -1$$

12. Consider the 2×2 matrix

$$\mathcal{K} = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}$$

(a) What are the eigenvalues of $\mathcal{K}$?

(b) What are the eigenvectors of $\mathcal{K}$?

$$K = \left(\frac{A+B}{2}\right)\sigma_z + \left(\frac{A+B}{2}\right)\mathbb{1} + C_R\sigma_x + C_I\sigma_y$$

eigenvalues are

$$\lambda_\pm = \frac{A+B}{2} \pm \sqrt{\left(\frac{A+B}{2}\right)^2 + C_R^2 + C_I^2}$$

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix}\begin{pmatrix} 1 \\ u_\pm \end{pmatrix} = \begin{pmatrix} A + C^* u_\pm \\ C + B u_\pm \end{pmatrix} = \begin{pmatrix} \lambda_\pm \\ \lambda_\pm u_\pm \end{pmatrix}$$

$$A + C^* u_\pm = \lambda_\pm$$

$$u_\pm = \frac{\lambda_\pm - A}{C^*}$$

normalized eigenvectors

$$= \begin{pmatrix} 1 \\ \frac{\lambda_\pm - A}{C^*} \end{pmatrix} \frac{1}{\sqrt{1 + \frac{(\lambda_\pm - A)^2}{|C|^2}}}$$

13. A beam of light with wavelength 660 nm is sent along the z axis through a polaroid filter that passes only x polarized light. The beam is initially polarized at 30° to the x axis, and the total energy of the pulse is exactly 10 Joules. Estimate the fluctuations of the energy of the transmitted beam, $\langle (E-\bar{E})^2\rangle^{1/2}$. Express the fluctuations as a fraction of the average transmitted energy. (Hint: Consider the binomial distribution, with N tries with probability p of success of each try.)

$$a = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$|\langle a|x\rangle|^2 = P_x = \frac{3}{4} = \text{ave trans. energy}$$

$$N_\gamma = \frac{E_{tot}}{\hbar c/\lambda} = \#\text{ tries}$$

$$P(n) = \frac{p_x^{\,n}(1-p_x)^{N-n}\,N!}{(N-n)!\,n!}$$

$$\langle n\rangle = \sum_n P(n)\,n = \sum_n \frac{p_x^{\,n-1}(1-p_x)^{(N-1)-(n-1)}}{(N-1-(n-1))\,(n-1)!}(N-1)!\,p_x N$$

$$= p_x N_\gamma$$

$$\langle n(n-1)\rangle = p_x^2 N_\gamma(N_\gamma-1)$$

$$\text{Fluctuation} = \langle (n-\langle n\rangle)^2\rangle$$

$$= \langle n^2\rangle - \langle n\rangle^2 = \langle n(n-1)\rangle + \langle n\rangle - \langle n\rangle^2$$

$$= p_x^2 N_\gamma(N_\gamma-1) + p_x N_\gamma - p_x^2 N_\gamma^2$$

$$= (p_x - p_x^2)N_\gamma = p_x(1-p_x)N_\gamma$$

$$\langle (E-\bar{E})^2\rangle^{1/2} = \frac{\hbar c}{\lambda}\left[p_x(1-p_x)N_\gamma\right]^{1/2} = \sqrt{E_{TOT}\frac{\hbar c}{\lambda}}\cdot\left(p_x(1-p_x)\right)^{1/2}$$

14. Considering light moving along the z axis and using the following definitions for $|R\rangle$ and $|L\rangle$ in terms of x and y polarized light,

$$|R\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \quad |L\rangle \equiv \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle),$$

(a) In terms of $|R\rangle$ and $|L\rangle$ write the states $|45\rangle$ and $|135\rangle$ which are linearly polarized at 45° and 135° relative to the x axis.

(b) Calculate the 2×2 transformation matrix from the $45, 135$ basis to the RL basis.

(c) Show that this transformation is unitary.

$$|45\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$$

$$= \frac{1}{\sqrt{2}}\left\{ \frac{1}{\sqrt{2}}\left[|R\rangle + |L\rangle\right] - i\frac{1}{\sqrt{2}}\left[|R\rangle - |L\rangle\right]\right\}$$

$$= \frac{(1-i)}{2}|R\rangle + \frac{(1+i)}{2}|L\rangle$$

$$|135\rangle = -\frac{(1+i)}{2}|R\rangle - \left(\frac{1-i}{2}\right)|L\rangle$$

$$U = \begin{pmatrix} \langle 45|R\rangle & \langle 135|R\rangle \\ \langle 45|L\rangle & \langle 135|L\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1+i}{2}\right) & -\left(\frac{1-i}{2}\right) \\ \left(\frac{1-i}{2}\right) & -\frac{(1+i)}{2} \end{pmatrix}$$

$$UU^{+} = \frac{1}{4}\begin{pmatrix} (1+i) & -(1-i) \\ (1-i) & -(1+i) \end{pmatrix}\begin{pmatrix} (1-i) & -(1+i) \\ (1+i) & -(1-i) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (2+2) & (2-2) \\ (2-2) & (2+2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

15. The probability that a photon in state $|\Psi\rangle$ passes through an x-polaroid is the average value of a physical observable which might be called the *x-polarizedness*.

(a) Write down the operator $\boldsymbol{P}_x$ corresponding to the observable as a matrix in the $\boldsymbol{xy}$ representation. $\langle\Psi|\boldsymbol{P}_x|\Psi\rangle$ is the probability that $|\Psi\rangle$ makes it through the filter.

(b) What are it's eigenvalues and eigenstates?

(c) Write the matrix in the $\boldsymbol{RL}$ basis, and show that the eigenvalues are the same as in the $\boldsymbol{xy}$ basis.

a) $= |x\rangle\langle x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\lambda = 1, 0$, eigenvalues $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

c) $|x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$

$|x\rangle\langle x| = \frac{1}{2}|R\rangle\langle R| + \frac{1}{2}|R\rangle\langle L| + \frac{1}{2}|L\rangle\langle R| + \frac{1}{2}|L\rangle\langle L|$

In RL basis $= |R\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|x\rangle\langle x| = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\mathbb{1} + \sigma_x)\frac{1}{2}$

eigenvalues are $\frac{1 \pm 1}{2} = 1, 0$

16. The trace of a matrix $\boldsymbol{A}$ is defined as:

$$\mathrm{Tr}\boldsymbol{A} \equiv \sum_i \boldsymbol{A}_{ii}$$

(a) Show that the trace of $\boldsymbol{A}$ is invariant under a transformation of basis,

$$\boldsymbol{A} \to \boldsymbol{U}^\dagger \boldsymbol{A} \boldsymbol{U}$$

(b) Show that $\mathrm{Tr}\boldsymbol{AB} = \mathrm{Tr}\boldsymbol{BA}$.

$$\mathrm{Tr}\, U^\dagger A U = U^\dagger_{ij} A_{jk} U_{ki}$$

$$= U_{ki} U^\dagger_{ij} A_{jk} = A_{kk} = \mathrm{Tr} A \checkmark$$

$$\mathrm{Tr}\, AB = A_{ij} B_{ji} = B_{ji} A_{ij} = \mathrm{Tr}\, BA$$

17. A plane polarized photon at $\theta = 45°$ enters a special crystal with indices of refraction:
n_x=1.50 for photons polarized along the x axis
n_y=1.52 for photons polarized along the y axis.
Assuming the wavelength of the light is 660 nm before it enters the crystal, choose the thickness of the crystal such that the outgoing light is right circularly polarized. Assume the dispersion is linear, $k = n\omega/c$.

$$u = \frac{e^{-i\omega\tau}}{\sqrt{2}} \begin{pmatrix} e^{ik_x t} \\ e^{ik_y t} \end{pmatrix} = \frac{e^{ik_x - i\omega t}}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\Delta n\omega t/c} \end{pmatrix}$$

For RCP, $e^{i\Delta n \omega t/c} = i$

$$\frac{\pi}{2} = \Delta n \omega t/c$$

$$t = \frac{\pi c}{2\omega \Delta n}$$

18. Consider the matrix for rotation about the z axis,

$$R(\theta) = e^{-i\sigma_z\theta/2}.$$

Show that after rotation about the z axis,

$$R(\theta)\sigma_x R^{-1}(\theta) = \sigma_x\cos(\theta) + \sigma_y\sin(\theta)$$

$$R\,\sigma_x\,R^{-1} = \left(\cos\frac{\varphi}{2} - i\sigma_z\sin\frac{\varphi}{2}\right)\sigma_x\left(\cos\frac{\varphi}{2} + i\sigma_z\sin\frac{\varphi}{2}\right)$$

$$= \sigma_x\cos^2\varphi/2 + \sigma_z\sigma_x\sigma_z\sin^2\varphi/2 - i\left[\sigma_z,\sigma_x\right]\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$$

$$\sigma_z\sigma_x\sigma_z = -\sigma_x, \quad \left[\sigma_z,\sigma_x\right] = 2i\sigma_y$$

$$= \sigma_x\cos\varphi + \sigma_y\sin\varphi$$

19. Consider a basis for spin-up and spin-down electrons (along the z axis),

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

(a) Write down the 4 vectors describing an electron with spin pointed along the positive/negative directions of x and y axes.

(b) Write the six density matrices describing electrons polarized along the positive/negative directions of each of the three axes.

(c) Write the density matrix describing a mixture of 60% spin-up and 40% spin down.

(d) Using the density matrix, calculate $< y,+|S_z|y,+\rangle$.

(a)

$$|x\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |x\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \quad |y\uparrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \quad |y\downarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

(b)

$$\rho_{z\uparrow} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \rho_{z\downarrow} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\rho_{x\uparrow} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \quad \rho_{x\downarrow} = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$\rho_{y\uparrow} = \frac{1}{2}\begin{pmatrix}1 & -i\\i & 1\end{pmatrix}, \quad \rho_{y\downarrow} = \frac{1}{2}\begin{pmatrix}1 & i\\-i & 1\end{pmatrix}$$

c) $\rho_{60/40} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix}$

d) $\langle y\uparrow | S_z | y\uparrow \rangle$

$= \frac{1}{2} \mathrm{Tr}\, \rho_{y\uparrow} \sigma_z$

$= \frac{1}{2} \mathrm{Tr} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= 0$

20. Neutral Kaon Oscillations: There are two kinds of neutral kaons one can make using down and strange quarks,

$$|K^0\rangle = |d\bar{s}\rangle, \quad |\bar{K}^0\rangle = |s\bar{d}\rangle.$$

If particle and anti-particle symmetry (CP) were exact, the two species would have equal masses, and the Hamiltonian (for a kaon in the ground state of a well) would be

$$H_0 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

However, there is an additional term that mixes the states due to the weak interaction that mixes the two states,

$$H_m = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}.$$

The masses of a kaon neutral kaon are 497.6 MeV, without mixing, but after adding the mixing term the masses differ by 3μeV. The two eigen-states are known as K_S (K-short) and K_L (K-long), because they decay with quite different lifetimes.

(a) What is ϵ?

(b) If one creates a kaon in the K_0 state at time $t = 0$, find the probability it would be measured as a $\bar{K}^0$ as a function of time.

ⓐ $H = M\mathbb{1} + \epsilon\sigma_x$

$E_\pm = M \pm \varepsilon$, $\varepsilon = \frac{3}{2}\mu eV$

ⓑ $|K_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\bar{K}_0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = e^{-iMt} e^{-i\varepsilon\sigma_x t/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= e^{-iMt}\left[\cos\left(\frac{\epsilon t}{\hbar}\right) - i\sin\left(\frac{\epsilon t}{\hbar}\right)\sigma_x\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= e^{-iMt}\begin{pmatrix} \cos\frac{\epsilon t}{\hbar} \\ -i\sin\frac{\epsilon t}{\hbar} \end{pmatrix}$

$P_{K_0} = \cos^2\frac{\epsilon t}{\hbar}$

21. Neutrino Oscillations: There are three kinds of neutrinos corresponding to the three lepton families, and recent evidence has suggested that they may oscillate between generations. Here we consider two flavors, the μ neutrino and the τ neutrino. Suppose that the Hamiltonian can be written as a free term plus a term that mixes the μ and τ neutrinos, which is proportional to α.

$$\mathcal{H} = \begin{pmatrix} \sqrt{k^2+m_\mu^2} & 0 \\ 0 & \sqrt{k^2+m_\tau^2} \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) Supposing you are in the rest frame of the neutrino and that the momentum k is zero, show that the evolution operator $e^{i\mathcal{H}t}$ can be written as

$$-e^{i(m_\mu+m_\tau)t/2\,\hbar}\{\cos\omega t + i\sigma_n \sin\omega t\}\,,$$

where

$$\omega \equiv \sqrt{\alpha^2 + (\frac{m_\tau - m_\mu}{2})^2}\ /\hbar$$

$$\sigma_n \equiv \frac{\frac{m_\tau - m_\mu}{2}\sigma_z + \alpha\sigma_x}{\omega}$$

(b) If a neutrino starts as a μ neutrino, what is the probability, as a function of time, of being a τ neutrino?

(c) As a function of the masses and α, what is the oscillation time?

(d) If the neutrinos are extremely relativistic, $k >> m$, describe how the oscillation time translates into an oscillation as a function of the distance from the creation.

a) with $k=0$, $H = \left(\frac{m_\mu+m_\tau}{2}\right) + \left(\frac{m_\mu-m_\tau}{2}\right)\sigma_z + \alpha\sigma_x$

$H = \frac{m_\mu+m_\tau}{2} + \hbar\omega\,\sigma_n$, $(\hbar\omega)^2 = \left(\frac{m_\mu-m_\tau}{2}\right)^2 + \alpha^2$

$\sigma_n = \frac{\frac{m_\mu-m_\tau}{2}\sigma_z + \alpha\sigma_x}{\hbar\omega}$, $\sigma_n^2 = 1$

$U = e^{-iHt/\hbar} = e^{-i(m_\mu+m_\tau)t/\hbar}\{\cos\omega t - i\sigma_n \sin\omega t\}$

b) $P_{\mu\to\tau} = |\langle\tau|U|\mu\rangle|^2 = \frac{\alpha^2}{\hbar^2\omega^2}\sin^2\omega t$

c) $T_0 = \frac{\pi}{\omega} =$

d) $\tau = \gamma\tau_0$, $\gamma \simeq \frac{k}{m}$