1. Consider a two-dimensional zero-temperature non-relativistic gas of identical spin-up fermions whose mass is m. They are confined in a two-dimensional box of dimensions, L_x and L_y. The quantum numbers characterizing the single-particle eigenstates are n_x and n_y. The box is divided in half along the x axis. The eigenstates with odd n_x now disappear, while there are now two solutions (one for each half of the box) for each even value of n_x. Assuming the size of the box is large compared to the inverse Fermi momentum, find the penalty, expressed as an energy per unit length ($\Delta E/2L_y$), for dividing the box. Express your answer in terms of the Fermi momentum and the mass.

$$\Delta E = \sum_{\substack{k_y \text{ s.t.} \\ \frac{\hbar^2 k_y^2}{2m} < \varepsilon_f}} \frac{1}{2}\left(\varepsilon_f - \frac{\hbar^2 k_{\cancel{x}}^2}{2m}\right)$$

$$= \frac{\hbar^2}{4m} \sum_{k_y < k_f} \left(k_f^2 - k_y^2\right)$$

$$= \frac{\hbar^2}{4m} L_y \int_0^{k_f} \frac{dk_y}{\pi} \left(k_f^2 - k_y^2\right)$$

$$= \frac{\hbar^2}{4m\pi} L_y \left(k_f^3 - \frac{1}{3} k_f^3\right)$$

$$= \frac{1}{3\pi} L_y k_f \varepsilon_f$$

$$\frac{\Delta E}{2L_y} = \frac{k_f \varepsilon_f}{6\pi} = \frac{\hbar^2 k_f^3}{12 m \pi}$$

2. In the interior of a neutron star, the neutron-to-proton ratio is very high. This results despite the fact that the proton mass is 1.3 MeV higher than the neutron mass. This occurs because protons must balanced by an equal number of electrons. Furthermore, protons and neutrons may be interchanged via the reaction,

$$p + e \leftrightarrow n$$

(We have neglected the neutrinos in this reaction because they are free to leave the star due to their massless nature.)

The masses of the particles are:

$$m_p c^2 = 938.27 \text{ MeV}, \quad m_n c^2 = 939.57 \text{ MeV}, \quad m_e c^2 = 0.511 \text{ MeV}$$

For the problems below, you may assume the protons and neutrons are non-relativistic, $E = mc^2 + (\hbar ck)^2/(2mc^2)$, but the electrons must be treated as relativistic particles, $E = [(\hbar ck)^2 + (mc^2)^2]^{1/2}$. It is useful to remember that $\hbar c = 197.326$ MeV·fm.

(a) If the baryon(neutrons or protons) density equals n_B, express the corresponding constraint involving the Fermi momenta of the protons and neutrons.

(b) From the constraint that the system is electrically neutral, describe how the proton's Fermi momentum is related to the electron's Fermi momentum.

(c) Describe how minimizing the overall energy results in a constraint involving the three Fermi momenta.

(d) For 4 cases, $n_B = \cancel{0.0001}$, $0.\cancel{0}01$, 0.1, 1.0 baryons per cubic Fermi, solve the expressions above and plot the neutron/proton ratio as a function of n_B. Solving the three equations may involve finding roots numerically.

0.001 0.01, 0.1, 1.0

(a) $n_n = \frac{2 \cdot 4\pi}{3} \frac{1}{(2\pi\hbar)^3} p_n^3 = \frac{1}{3\pi^2}\left(\frac{p_n}{\hbar}\right)^3$

$n_p = \frac{1}{3\pi^2}\left(\frac{p_p}{\hbar}\right)^3 = n_e$ thus $p_p = p_e$

① $m_p + \frac{p_p^2}{2M_p} + \sqrt{m_e^2 + p_p^2} = m_n + p_n^2/2m_n$

② $n_B = \frac{1}{3\pi^2\hbar^3}\left(p_p^3 + p_n^3\right)$

2 eq.s $\leftrightarrow$ 2 unknowns (p_p & p_n)

(b) $p_p = p_e$

(c) Solve the 2 eq.s 2 unknowns

(d) To solve for p_p,

$m_p - m_n + \frac{p_p^2}{2m_p} + \sqrt{p_p^2 + m_e^2} = \left(3\pi^2\hbar^3 n_B - p_p^3\right)^{2/3} / 2m_n$

Solving these numerically for p_p

$$p_p = \begin{cases} 1.65 \text{ MeV/c} &, n_B = 0.0001 \text{ fm}^{-3} \\ 3.24 \text{ MeV/c} &, n_B = 0.001 \text{ fm}^{-3} \\ 16.4 \text{ MeV/c} &, n_B = 0.01 \text{ fm}^{-3} \\ 42.9 \text{ MeV/c} &, n_B = 0.1 \text{ fm}^{-3} \\ 179 \text{ MeV/c} &, n_B = 1.0 \text{ fm}^{-3} \end{cases}$$

$$p_n^3 = 3\pi^2 \hbar^3 n_B - p_p^3$$

$$\frac{n_n}{n_p} = \frac{p_n^3}{p_p^3} = \begin{cases} 5078 &, n_B = 0.0001 \text{ fm}^{-3} \\ 6704 &, n_B = 0.001 \text{ fm}^{-3} \\ 2003 &, n_B = 0.01 \text{ fm}^{-3} \\ 286 &, n_B = 0.1 \text{ fm}^{-3} \\ 38.6 &, n_B = 1.0 \text{ fm}^{-3} \end{cases}$$

3. **Correlation/anticorrelation in a quantum gas:**
Consider a uniform gas of non-interacting spin-half particles in the ground state. The wave function may be written

$$|\phi\rangle = \prod_{\alpha, |k_\alpha| < k_f} a^\dagger_\alpha |0\rangle,$$

where the product includes all states α with momentum $k_\alpha < k_f$ and spin s_α. The density-density correlation function is defined as

$$C_{s_1,s_2}(\mathbf{x}_2 - \mathbf{x}_1) \equiv \frac{\langle\phi|\Psi^\dagger_{s_1}(\mathbf{x}_1)\Psi^\dagger_{s_2}(\mathbf{x}_2)\Psi_{s_2}(\mathbf{x}_2)\Psi_{s_1}(\mathbf{x}_1)|\phi\rangle}{\langle\phi|\Psi^\dagger_{s_1}(\mathbf{x}_1)\Psi_{s_1}(\mathbf{x}_1)|\phi\rangle\langle\phi|\Psi^\dagger_{s_2}(\mathbf{x}_2)\Psi_{s_2}(\mathbf{x}_2)|\phi\rangle}$$

This function expresses the correlation of two particles with spin s_1 and s_2 being separated by $\mathbf{x}_2 - \mathbf{x}_1$. It is defined in such a way that it is unity if the probability of seeing two particles at $\mathbf{x}_1$ and $\mathbf{x}_2$ is the product of the probabilities of observing each particle independently.

(a) Show that the density-density correlation function can be written as

$$C_{s_1,s_2}(\mathbf{x}_2 - \mathbf{x}_1) = \frac{\sum_{\alpha,\beta}\left(1 - \delta_{s_1,s_2} e^{i(\mathbf{k}_\alpha - \mathbf{k}_\beta)\cdot(\mathbf{x}_2 - \mathbf{x}_1)}\right)}{\sum_{\alpha,\beta}},$$

where the sums are over all momentum states with $k_\alpha < k_f$ and $k_\beta < k_f$.

(b) By changing the sum over states to a three-dimensional integral over $\mathbf{k}$, find an analytic expression for the density-density correlation function in terms of the Fermi momentum k_f.

(a) $\Psi_{s_1}(x_2)\Psi_{s_2}(x_1)\left(\prod_{\gamma,\, k_\gamma < k_f} a^\dagger_\gamma\right)|0\rangle$

$$= \sum_{\alpha\beta,\, k_\alpha, k_\beta < k_f} e^{ik_\alpha x_1 + ik_\beta x_2}\,\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta} \prod_{\gamma\neq\alpha,\beta,\, k_\gamma<k_f} a^\dagger_\gamma |0\rangle$$

$\langle 0|\left(\prod_\gamma a_\gamma\right)\Psi^\dagger(x_1)\Psi^\dagger(x_2)$

$$= \sum_{\alpha',\beta',\, k_{\alpha'}, k_{\beta'} < k_f} e^{-ik_{\alpha'}x_1 - ik_{\beta'}x_2}\,\langle 0|\left(\prod_{\gamma\neq\alpha,\beta,\, k_\gamma<k_f} a_\gamma\right)\delta_{s_1 s_{\alpha'}}\delta_{s_2 s_{\beta'}}$$

$$= \sum_{\alpha\beta,\, k_\alpha, k_\beta < k_f}\Big[e^{ik_\alpha x_1 + ik_\beta x_2 - ik_\alpha x_1 - ik_\beta x_2}\left(\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta}\right)^2 - e^{ik_\alpha x_2 + ik_\beta x_1 - ik_\alpha x_1 - ik_\beta x_2}\,\delta_{s_1 s_\alpha}\delta_{s_2 s_\beta}\cdot\delta_{s_1 s_\beta}\delta_{s_2 s_\alpha}\Big]$$

$$= \sum_{\substack{\alpha,\beta \\ \text{s.t. } k_\alpha, k_\beta < k_f \\ s_\alpha = s_1,\, s_\beta = s_2}} - \delta_{s_1 s_2}\sum_{\substack{\alpha,\beta \\ \text{s.t. } k_\alpha, k_\beta < k_f \\ s_\alpha = s_\beta = s_1 = s_2}} e^{i(k_\alpha - k_\beta)(x_2 - x_1)}$$

(a) $C_{s_1 s_2}(\vec{x}_2 - \vec{x}_1)$

$$= \frac{\sum_{\substack{s_\alpha = s_1, s_\beta = s_2 \\ k_\alpha, k_\beta < k_f}} \left(1 - \delta_{s_1 s_2} e^{i(\vec{k}_\alpha - \vec{k}_\beta)\cdot(\vec{x}_1 - \vec{x}_2)}\right)}{\sum_{\substack{\alpha\beta, k_\alpha, k_\beta < k_f \\ s_\alpha = s_1, s_\beta = s_2}}}$$

(b) $\sum_\alpha \to V \int \frac{d^3 k_\alpha}{(2\pi)^3}$

$$C(x_1, x_2) = \frac{\frac{1}{(2\pi)^6} \int_{k_\alpha, k_\beta < k_f} d^3 k_\alpha d^3 k_\beta \left(1 - \cos (k_\alpha - k_\beta)\cdot(x_1 - x_2)\, \delta_{s_1 s_2}\right)}{\left(\frac{1}{(2\pi)^3} \frac{4\pi}{3} k_f^3\right)^2}$$

$$= 1 - \delta_{s_1 s_2} \left(\frac{4\pi}{3} k_f^3\right)^{-2} \cdot \int d^3 k_\alpha d^3 k_\beta \left[\cos k_\alpha \cdot \Delta x \cos k_\beta \cdot \Delta x + \sin \cancel{k_\alpha} \cdot \Delta x \sin \cancel{k_\beta} \cdot \Delta x\right]$$

$$= 1 - \left(\frac{4\pi}{3} k_f^3\right)^{-2} I^2 \delta_{s_1 s_2}$$

$$I = \int d^3 k \cos(\vec{k} \cdot \Delta \vec{X}) \quad \text{depends only on } |\Delta \vec{X}|$$

$$I(|\Delta x|) = 4\pi \int_0^{k_f} k_\perp d k_\perp \int_0^{(k_f^2 - k_\perp^2)^{1/2}} d k_z \cos k_z |\Delta \vec{X}|$$

$$= 4\pi \int_0^{k_f} k_\perp d k_\perp \frac{1}{|\Delta \vec{X}|} \sin\left(\sqrt{k_f^2 - k_\perp^2}\, \Delta X\right)$$

$$u = \sqrt{k_f^2 - k_\perp^2}$$

$$du = \frac{-1}{u} k_\perp d k_\perp$$

$$I(|\Delta x|) = 4\pi \int_0^{k_f} \frac{u\, du}{|\Delta \vec{X}|} \sin(u |\Delta \vec{X}|)$$

$$I(|\Delta x|) = 4\pi \int_0^{k_f} \frac{u\,du}{|\Delta \vec{x}|} \sin(u|\Delta \vec{x}|)$$

$$= \frac{4\pi}{|\Delta \vec{x}|^2} \cdot \int_0^{k_f} du \cos(u|\Delta \vec{x}|) - \frac{4\pi}{|\Delta \vec{x}|^2} u \cos(u|\Delta \vec{x}|) \Big|_0^{k_f}$$

$$= \frac{4\pi}{|\Delta \vec{x}|^3} \sin(k_f|\Delta \vec{x}|) - \frac{4\pi}{|\Delta \vec{x}|^2} k_f \cos(k_f|\Delta \vec{x}|)$$

$$C(\vec{x}_1 - \vec{x}_2) = 1 - \delta_{s_1 s_2} \left[\frac{\frac{1}{\Delta x^3} \sin(k_f \Delta x) - \frac{k_f}{\Delta x^2} \cos(k_f \Delta x)}{k_f^3/3} \right]^2$$

where $\Delta x = |\vec{x}_1 - \vec{x}_2|$

Note that

$$C(\Delta x = 0) = 0$$

4. Again, consider a non-interacting quantum gas of particles of mass m, with the ground state being expressed as

$$|\phi\rangle = \prod_{\alpha, |k_\alpha| < k_f} a_\alpha^\dagger |0\rangle,$$

where the product includes all states α with momentum $k_\alpha < k_f$. For this problem, ignore the spin indices. Consider an interaction between the particles of the form,

$$H_{\text{int}} = \frac{1}{2}\int d^3r_1 d^3r_2 V(\mathbf{r}_1 - \mathbf{r}_2)\Psi^\dagger(\mathbf{r}_1)\Psi^\dagger(\mathbf{r}_2)\Psi(\mathbf{r}_2)\Psi(\mathbf{r}_1)$$
$$V(\mathbf{r}_1 - \mathbf{r}_2) = \beta\delta(\mathbf{r}_2 - \mathbf{r}_1)$$

Find the first-order perturbative correction for the energy, $\langle\phi|H_{\text{int}}|\phi\rangle$, for particles in the gas in terms of k_f.

From previous problem

$$\langle H_{int}\rangle = \frac{1}{2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1 - \vec{r}_2) \cdot \langle 0| \prod_\gamma a_\gamma \, \psi(\vec{r}_1)\psi(\vec{r}_2)\psi^\dagger(\vec{r}_2)\psi(\vec{r}_1) \prod_\gamma a_\gamma^\dagger |0\rangle$$

$$= \frac{1}{2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1 - \vec{r}_2) \, C(\vec{r}_1 - \vec{r}_2) \cdot n^2$$

$$n = \frac{1}{(2\pi)^3}\int_{k<k_f} d^3k = \frac{1}{(2\pi)^3}\frac{4\pi}{3}k_f^3 = \frac{k_f^3}{6\pi^2}$$

$$\langle H_{int}\rangle = \frac{V}{2}\int d^3\Delta r \, V(\Delta\vec{r}) \, C(\Delta\vec{r}) \cdot \left(\frac{k_f^3}{6\pi^2}\right)^2$$

$$= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2 \int d^3r \, V(\vec{r}) \cdot \left\{ 1 - \left[\frac{\frac{1}{r^3}\sin k_f r - \frac{k_f}{r^2}\cos k_f r}{k_f^3/3} \right]^2 \right\}$$

$$= 0 \quad \text{because} \quad V \sim \delta(\vec{r}) \text{ \& } C(\vec{r}=0) = 0!$$