1. Consider the two electron holes in the p-shell of a neutral oxygen atom.

   (a) What is the $L - S - J$ of the ground state.

   (b) If the atom is in a magnetic field of 0.01 Tesla, find the magnetic energies of the originally degenerate $2J + 1$ states.

(a) Consider 2 holes

$S = 0, 1$

$S = 1$ is lowest Hund's Rule #1

$L = 0, 1, 2,$

is lowest Hund's Rule #2

BUT for permutation symmetry

orbital $L = 0, 2$ symmetric, $L = 1$ anti-symm.

spin $S = 0$ anti-symm., $S = 1$ anti-symm

Thus to be anti-symmetric

$L = 1.$

$J = 2$ to have highest J

$$\Delta E = -g \frac{e\hbar B}{2mc} M_J \qquad (12.29)$$

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

$$\Delta E = -g \frac{e\hbar B}{2mc} M_J$$

$$g = 1 + \frac{6 + 2 - 2}{12} = \frac{3}{2}$$

$$\Delta E = -\frac{3}{2} \frac{e\hbar}{2mc} B = -\frac{3}{2} 5.788 \cdot 10^{-5} \cdot 0.01$$

$$= 8.68 \cdot 10^{-7} \text{ eV}$$

2. One electron moves in a one-dimensional system and feel the interaction of two atoms. Approximate the interaction between the electrons and the atoms with the potential

$$V(x-R) = -\beta\delta(x-R),$$

where R is the position of an atom. Use the adiabatic approximation to

(a) Given the two atoms are separated by a distance r, find a transcendental equation relating k and r where the electronic binding energy is $\hbar^2k^2/(2m)$.

(b) Find the potential between the two atoms at small r,

$$V(r\to 0) \sim V(r=0) - \alpha r,$$

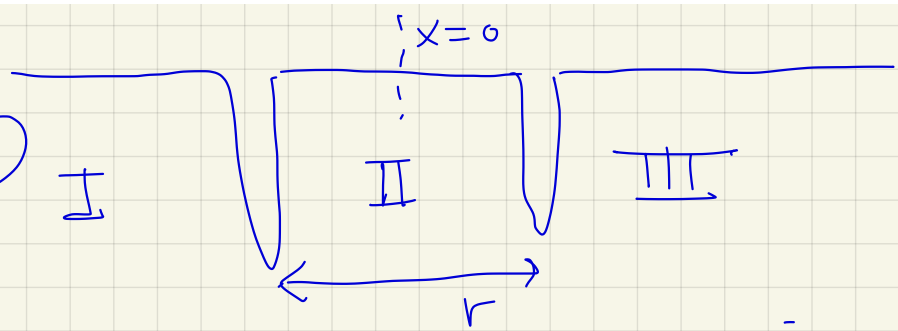
that is, find $V(r=0)$ and α. Do this by expanding the transcendental equation in terms of r. Hint: First, find $V(r=0)$ by solving the transcendental equation with $r=0$. Take derivatives of the transcendental equation with respect to r, then solve for dk/dr at $r=0$, and finally find dE/dr to obtain α.

(c) Find the potential between the two atoms at large r,

$$V(r\to\infty) = -\gamma\exp(-2k_\infty r),$$

that is, find γ. Hint: Use first order perturbation theory, assuming the unperturbed wave function is the bound state of one well, and the perturbation is the interaction with the second well.

(a)



$$\psi_I = Ae^{k(x+\frac{r}{2})}, \quad \psi_{II} = \cosh kx, \quad \psi_{III} = Ae^{-k(x-\frac{r}{2})}$$

$$A = \cosh\left(k\frac{r}{2}\right)$$

$$-kA = k\sinh\left(k\frac{r}{2}\right) = 2m\beta/\hbar^2 A$$

$$-kA + \frac{2m\beta}{\hbar^2}A = k\sinh\left(k\frac{r}{2}\right)$$

$$k\tanh\left(\frac{kr}{2}\right) = -k + 2m\beta/\hbar^2$$

$$\tanh\left(\frac{kr}{2}\right) = -1 + \frac{2m\beta}{\hbar^2 k}$$

(b) $\tanh \frac{kr}{2} = \frac{2m\beta}{\hbar^2 k} - 1$

At $r=0$, $k = \frac{2m\beta}{\hbar^2}$, $E = -\left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{\hbar^2}{2m}$

$$\frac{d}{dr} \tanh \frac{kr}{2} = \frac{d}{dr}\left(\frac{2m\beta}{\hbar^2 k} - 1\right)$$

$$\frac{k}{2} = -\frac{2m\beta}{\hbar^{\cancel{4}2} k^2} \frac{dk}{dr}$$

$$\frac{dk}{dr} = \frac{-\hbar^2 k^3}{\cancel{2}4m\beta} = -\frac{2m^2\beta^2}{\hbar^4}$$

$$E_B \simeq E(r=0) + \frac{dE}{dk}\frac{dk}{dr} \cdot r$$

$$=$$

(c) 

$$\psi = \sqrt{k}\, e^{-k|r|}$$

$$V = \int \psi^*(r)\, \psi(r)\, dr \; -\beta\, \delta(r - r_0)$$

$$= -\beta k e^{-2kr_0}$$

$$V(r) = -\frac{2m\beta^2}{\hbar^2} e^{-2kr}$$

4. Consider a particle of mass M and charge e moving in the $x-y$ plane under the influence of a magnetic field in the z direction. Ignore motion in the z direction.

(a) Show that the vector potential,

$$\vec{A} = \frac{B}{2}\left(x\hat{y} - y\hat{x}\right),$$

describes a magnetic field in the z direction.

(b) Express the vector potential in cylindrical coordinates, that is in terms of r, ϕ, $\hat{r}$ and $\hat{\phi}$.

(c) Write the Schrödinger equation,

$$\frac{\left(\vec{p} - e\vec{A}/c\right)^2}{2m}\psi(r,\phi) = E\psi(r,\phi), \qquad (12.71)$$

in cylindrical coordinates.

(d) Show that L_z commutes with the Hamiltonian.

(e) Assuming the solution is an eigenstate of L_z with eigenvalue $m\hbar$,

$$\psi(r,\phi) = e^{im\phi}\xi_m(r),$$

rewrite the Schrödinger equation for $\xi_m(r)$.

(f) **Extra Credit**: Solve for the $\xi_m(r)$ and the eigenenergies for the case where $m = 0$.

(a) $(\nabla \times \vec{A})_z = \partial_x A_y - \partial_y A_x = \frac{B}{2} + \frac{B}{2} = B$

$(\nabla \times \vec{A})_x = \partial_y A_z - \partial_z A_y = 0$

$(\nabla \times \vec{A})_y = \partial_z A_x - \partial_x A_z = 0$

(b) $\vec{A} = \frac{B}{2} r \hat{\varphi}$

(c) $\vec{\nabla} = \hat{z}\partial_z + \hat{r}\partial_r + \frac{\hat{\varphi}}{r}\partial_\varphi$

$\nabla^2 = \partial_z^2 + \partial_r^2 + \frac{1}{r^2}\partial_\varphi^2 + \frac{1}{r}\partial_r$

$E\psi = -\frac{\hbar^2}{2M}\nabla^2\psi - \frac{ie\hbar\vec{A}\cdot\vec{\nabla}\psi}{mc} - i\frac{\hbar e}{2Mc}\vec{\nabla}\cdot\vec{A} + \frac{e^2|\vec{A}|^2}{2Mc^2}$

$= -\frac{\hbar^2}{2M}\left(\partial_z^2 + \partial_r^2 + \frac{1}{r^2}\partial_\varphi^2 + \frac{1}{r}\partial_r\right) - \frac{ie\hbar B}{2Mc}\partial_\varphi + \frac{e^2B^2r^2}{8Mc^2}$

$$\left\{-\frac{\hbar^2}{2m}\left\{\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2} + \partial_z^2\right\} + \frac{m e \hbar B}{2Mc} + \frac{e^2B^2 r^2}{8Mc^2}\right\}\psi = E\psi$$

f) $\psi = e^{ik_z z} e^{im\varphi}\xi_m(r)$

Set $m = 0$

$$\xi_m(r) = e^{-r^2/2\sigma^2} \Leftarrow \text{GUESS}$$

$$\partial_r \xi(r) = e^{-r^2/2\sigma^2}\left(\frac{-r}{\sigma^2}\right)$$

$$\partial_r^2 \xi(r) = e^{-r^2/2\sigma^2}\left(-\frac{1}{\sigma^2} + \frac{r^2}{\sigma^4}\right)$$

$$-\frac{\hbar^2}{2M}\left\{-\frac{1}{\sigma^2} + \frac{r^2}{\sigma^4} - \frac{1}{\sigma^2}\right\} + \frac{e^2B^2r^2}{8Mc^2} = E$$

$$\frac{\hbar^2}{2M\sigma^4} = \frac{e^2B^2}{8Mc^2}, \quad \sigma^2 = \sqrt{\frac{4\hbar^2c^2}{e^2B^2}} = \frac{2\hbar c}{eB}$$

$$E = \frac{\hbar}{m}\left(\frac{eB}{2c}\right) = \frac{e\hbar B}{2mc}$$

$$= \frac{1}{2}\hbar\omega, \quad \omega = \frac{eB}{mc}$$

5. Consider a surface with 10 electrons per $\mu\mathrm{m}^2$. At what magnetic field (in Tesla) do you find the first dip in conductivity due to the quantum Hall effect?

$$n = 10^{13}$$

$$m = \frac{2\pi\hbar c n}{eB}$$

$$B/c = \frac{2\pi\hbar n}{e} = 0{,}0414 \quad \mathrm{T}$$