1. To show why derivatives are defined as shown in Eq. (13.12), show that

$$\partial_\mu x^2 = 2x_\mu, \text{ and } \partial^\mu x^2 = 2x^\mu,$$

where $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

For $\partial_\mu = \frac{\partial}{\partial x^\mu}$

$$\partial/\partial t \, (t^2 - |\vec{x}|^2) = 2t = x_{\mu=0}$$

$$\partial/\partial x_i \, (t^2 - |\vec{x}|^2) = -2x_i = 2x_{\mu=i}$$

For $\partial^\mu = -\frac{\partial}{\partial x_\mu}$

$$\partial^0 (t^2 - |\vec{x}|^2) = 2t = x^0$$

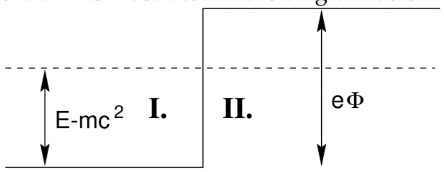
$$\partial^{\mu=i} (t^2 - |\vec{x}|^2) = -\frac{\partial}{\partial x_{i=\mu}} (t^2 - |\vec{x}|^2)$$

$$= 2x^{(\mu=i)}$$

2. Consider a charged relativistic particle interacting with the electromagnetic field, and described by the Klein-Gordon equation.

$$\left[(i\hbar\partial_t - e\Phi)^2 + c^2\hbar^2\partial_x^2 - m^2c^4\right]\psi(x,t) = 0$$

The electrostatic potential Φ is illustrated in the diagram below.



Consider a solution for a particle incident from the left,

$$\psi_I(x,t) = e^{(-iEt+ikx)/\hbar} + Be^{(-iEt-ikx)/\hbar}$$
$$\psi_{II}(x,t) = Ce^{(-iEt+ik'x)/\hbar},$$

where $E = \sqrt{m^2c^4 + k^2}$.

Calculate the charge and current densities (include direction) in regions I and II for each of the following three cases.

(a) $e\Phi < E - mc^2$.
(b) $E - mc^2 < e\Phi < E + mc^2$.
(c) $e\Phi > E + mc^2$.

a) BC. $1 + B = C$

$k(1-B) = k'C$

$k' = \sqrt{(E-e\Phi)^2 - m^2}$

$E - e\Phi > mc^2$, $k' =$ real

$C\left(1 + \frac{k'}{k}\right) = 2$

$C = 2k/(k+k')$

$B = k - k'/(k+k')$

$$\rho_I = E\left|e^{ikx} + \frac{k-k'}{k+k'}e^{-ikx}\right|^2$$
$$= E\left\{1 + \frac{(k-k')^2}{(k+k')^2} + 2\frac{(k-k')}{k+k'}\cos 2kx\right\}$$

$$j_I = \frac{1}{2}\left(e^{-ikx} + \frac{k-k'}{k+k'}e^{ikx}\right)\left(-i\partial_x\left(e^{ikx} + \left(\frac{k-k'}{k+k'}\right)e^{-ikx}\right)\right) + \frac{1}{2}\left(i\partial_x\left(e^{-ikx} + \frac{k-k'}{k+k'}e^{ikx}\right)\right)\left(e^{ikx} + \frac{k-k'}{k+k'}e^{-ikx}\right)$$
$$= \frac{k}{2}\left\{1 - \left(\frac{k-k'}{k+k'}\right)^2 + 2i\left(\frac{k-k'}{k+k'}\right)\sin 2kx\right\} + \frac{k}{2}\left\{1 - \left(\frac{k-k'}{k+k'}\right)^2 - 2i\left(\frac{k-k'}{k+k'}\right)\sin 2kx\right\}$$
$$= k\left\{1 - \left(\frac{k-k'}{k+k'}\right)^2\right\} = \frac{4k^2k'}{(k+k')^2}$$

$$\rho_{II} = (E - e\Phi)\frac{4k^2}{(k+k')^2}$$

$$j_{II} = k'\frac{4k^2}{(k+k')^2}$$

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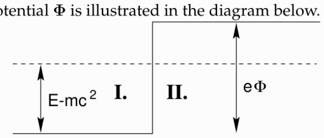

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(a) $e\Phi < E - mc^2$.

(b) $E - mc^2 < e\Phi < E + mc^2$.

(c) $e\Phi > E + mc^2$.

(b) $k' = \sqrt{(E - e\Phi)^2 - m^2} =$ imaginary

$\Psi_{II} = C e^{-|k'|x}$

B.C.

$1 + B = C$

$k(1 - B) = i|k'|C$

$C = \dfrac{2k}{k + i|k'|}$

$B = \dfrac{k - i|k'|}{k + ik'}, \quad B^* B = 1$

$\rho_I = E\left|e^{ikx} + Be^{-ikx}\right|^2 = E\left\{2 + Be^{-2ikx} + B^* e^{ikx}\right\}$

$= E\left\{2 + 2B_R \cos 2kx + 2B_I \sin 2kx\right\}$

$j_I = \frac{1}{2}\left(e^{-ikx} + B^* e^{ikx}\right)(-i\partial_x)\left(e^{ikx} + Be^{-ikx}\right) + h.c.$

$= \frac{1}{2}\left(k - k + kB^* e^{2ikx} - kBe^{-2ikx}\right) + h.c.$

$= \emptyset$

$\rho_{II} = \dfrac{4k^2}{k^2 + |k'|^2}(E - e\Phi)e^{-2|k'|x}$

$j_{II} = \dfrac{i}{2}\dfrac{4k^2}{k^2 + |k'|^2}\left\{ike^{-2|k'|x} + h.c.\right\} = 0$

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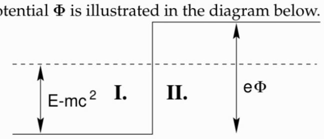

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(a) $e\Phi < E - mc^2$.

(b) $E - mc^2 < e\Phi < E + mc^2$.

(c) $e\Phi > E + mc^2$.

(c) $k' = \sqrt{(E - e\Phi)^2 - m^2} = \text{real}$

SAME as A ✓

$$\rho_I = E\left\{1 + \frac{(k-k')^2}{(k+k')^2} + 2\frac{(k-k')}{k+k'}\cos 2kx\right\}$$

$$j_I = \frac{4k^2k'}{(k+k')^2}$$

$$\rho_{II} = (E - e\Phi)\frac{4k^2}{(k+k')^2}$$

$$j_{II} = k'\frac{4k^2}{(k+k')^2}$$

Note that density in II is negative!

3. Consider the same case as above, except with no electrostatic potential. Instead, consider a different mass in region I and region II, with $m_{II} > m_I$. For each of the following two cases, calculate the charge and current densities in regions I and II.

   (a) $E > m_{II}c^2$
   (b) $E < m_{II}c^2$

I $\qquad$ II

$E^2 = k^2 + m^2$ $\qquad$ $E^2 = k'^2 + (m + \Phi)^2, \quad \Phi = m_{II} - m_I$

$k' = \sqrt{E^2 - (m+\Phi)^2}$

$\Psi_I = e^{ikx} + Be^{-ikx}, \quad \Psi_{II} = Ce^{ik'x}$

a) k' is real, just like A, C from previous problem.

$C = 2k/(k+k')$

$B = k - k'/(k+k')$

$\rho_I = E\left\{1 + \frac{(k-k')^2}{(k+k')^2} + 2\frac{(k-k')}{k+k'}\cos 2kx\right\}$

$j_I = 0$

b) k' is imaginary $\qquad \Psi_{II} = Ce^{-|k'|x}$

$C = \frac{2k}{k + i|k'|}, \quad B = \frac{k - i|k'|}{k + i|k'|}$

$\rho_{II} = |C|^2 E e^{-2kx} = \frac{4k^2}{k^2 + |k'|^2} e^{-2kx}$

$j_{II} = 0$

4. Consider the Dirac representation,

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

and the chiral representation,

$$\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

The spinors, $u_\uparrow$ and $u_\downarrow$, represent positve-energy eigenvalues of the Dirac equation assuming the momentum is along the z axis.

$$(m\beta + p_z\alpha_z)\, u(p_z) = E u(p_z)\ ,$$

The spin labels, ↑ and ↓ refer to the positive and negative values of the spin operator,

$$\Sigma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

Write the four-component spinors $u_\uparrow$ and $u_\downarrow$ in terms of p, E and m :

(a) in the Dirac representation.
(b) in the chiral representation.
(c) in the limit $p_z \to 0$ for both representations.
(d) in the limit $p_z \to \infty$ for both representations.

$$\frac{(E-m)^2}{p^2} + 1 = \frac{2E^2 - 2mE}{p^2}$$

$$u_\uparrow = \left(\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix} \right) e^{(-iEt + ipz)/\hbar}$$

$$(m\beta + p\alpha_z)\, u_\uparrow = E u_\uparrow$$

a) Dirac: $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$

$$ma + pb = Ea \Rightarrow a = pb/(E-m)$$

$$-mb + pa = Eb \Rightarrow$$

$$u_\uparrow = \begin{pmatrix} p/(E-m) \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{normalizing} \quad \frac{p}{\sqrt{2E^2 - 2mE}} \begin{pmatrix} 1 \\ 0 \\ (E-m)/p \\ 0 \end{pmatrix}$$

$$u_\downarrow = \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} \qquad ma - pb = Ea \qquad a = \frac{-p}{E-m} b$$

$$u_\downarrow = \begin{pmatrix} 0 \\ -p/(E-m) \\ 0 \\ 1 \end{pmatrix} \Rightarrow \frac{p}{\sqrt{2E^2 - 2mE}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{(E-m)}{p} \end{pmatrix}$$

b) $\beta = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$

$$u_\uparrow = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

$$-mb + pa = Ea$$
$$-ma - pb = Eb$$
$$a = \frac{-m}{E-p}\, b$$

$$u_\uparrow = \begin{pmatrix} \frac{-m}{E-p} \\ 0 \\ 1 \\ 0 \end{pmatrix} \xRightarrow{\text{normalising}} \begin{pmatrix} -1 \\ 0 \\ (E-p)/m \\ 0 \end{pmatrix} \frac{m}{\sqrt{2E^2-2Ep}}$$

$$u_\downarrow = \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix}$$

$$-mb - pa = Ea$$
$$-ma + pb = Eb$$
$$a = \frac{-m}{E+p}\, b \qquad b = \frac{E+p}{m}\, a$$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\left(\frac{E+p}{m}\right) \end{pmatrix} \frac{m}{\sqrt{2E^2+2Ep}}$$

p

$$u_\downarrow = \begin{pmatrix} 0 \\ m/(E+p) \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ (E-p)/m \\ 0 \\ 1 \end{pmatrix} \frac{m}{\sqrt{2E^2-2Ep}}$$

(c) As $p \to 0$

Dirac: $u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Chiral: $u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$, $u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$

d) As $p \to \infty$

Dirac: $u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$, $u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$

Chiral $u_\uparrow = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $u_\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

5. Consider a solution to the Dirac equation for massless particles, $u_+(\vec{p})$, where the + denotes the fact that the solution is an eigenstate of the spin operator in the $\hat{p}$ directions,

$$(\vec{\Sigma} \cdot \hat{p}) u_+(\vec{p}, x) = u_+(\vec{p}, x).$$

Show that the operator β operating on $u_+(p)$ gives a negative energy solution but is still an eigenstate of $\vec{\Sigma} \cdot \hat{p}$ with eigenvalue $+1$.

$$H = \vec{\alpha} \cdot \vec{P}$$

$$\{\beta, H\} = 0$$

$$\{\beta, \vec{\Sigma} \cdot \vec{p}\} = p_i \{\beta, \alpha_i\} = 0$$

$$H \beta |\psi\rangle = -\beta H |\psi\rangle$$

$$= -E \beta |\psi\rangle$$

β switches energy

but

$$(\vec{\Sigma} \cdot \hat{p}) \beta |\psi\rangle = \beta |\vec{\Sigma} \cdot \hat{p}\rangle |\psi\rangle$$

β leaves $\vec{\Sigma} \cdot \hat{p}$ unchanged

6. Consider a massless spin half particle of charge e in a magnetic field in the $\hat{z}$ direction described by the vector potential

$$\vec{A} = Bx\hat{y}.$$

The Hamiltonian is then

$$H = \alpha_x(-i\hbar\partial_x) + \alpha_y(-i\hbar\partial_y - eBx).$$

(a) Show that the Hamiltonian commutes with $-i\hbar\partial_y$ and $i\hbar\partial_z$.

(b) The wave function can then be written as

$$\psi_{k_y,k_z}(x,y,z) = e^{ik_y y + ik_z z}\phi_{k_y,k_z}(x),$$

After setting $k_z = k_z = 0$, show that the energy can be found by solving the equation

$$E^2\phi_\pm(x) = (-\hbar^2\partial_x^2 + e^2B^2x^2 - e\hbar B\Sigma_z)\phi_\pm(x).$$

(c) Show that the eigen-values of the operator H^2 are

$$E_\pm^2 = (2n + 1 \mp 1)e\hbar B, \quad n = 0, 1, 2\cdot,$$

where the $\pm$ refers to eigenvalues of Σ_z. You can do this mapping to the harmonic oscillator and then using the solutions to the harmonic oscillator from Chapter 3. Note that when the the eigenvalue of Σ_z is $+1$, there exists a solution with $E = 0$.

a) $-i\hbar\partial_y$: By inspection, no "y" in H

$-i\hbar\partial_z$: " " " "z" in H

b) $H^2 = -\hbar^2\partial_x^2 + (-i\hbar\partial_y - eBx)^2 - \hbar^2\partial_z^2$

$-i\hbar\alpha_x\alpha_y eB$, $-i\alpha_x\alpha_y = \Sigma_z$

$-i\hbar\partial_y \to \hbar k_y \to 0$, $-i\hbar\partial_z \to \hbar k_z = 0$

$H^2 = -\hbar^2\partial_x^2 + e^2B^2x^2 - e\hbar B\Sigma_z$

c) Map to H.O. · divide by $\frac{1}{2M}$

$$\frac{H^2}{2m} = -\frac{\hbar^2}{2M}\partial_x^2 + \frac{1}{2}M\frac{e^2B^2}{M^2}x^2 - \frac{e\hbar B}{2M}\Sigma_z$$

Looks like H.O with $\omega = \frac{eB}{M}$, + Σ_z term

$$\frac{E^2}{2M} = \left(n + \frac{1}{2}\right)\frac{e\hbar B}{M} \mp \frac{e\hbar B}{2M}$$

$$E^2 = ((2n+1) \mp 1)e\hbar B$$

7. Using the definitions for α and β in Eq. (13.73) show that

a) show that

$$b_k^\dagger b_k - d_{-k}^\dagger d_{-k} = \alpha^\dagger \alpha - \beta^\dagger \beta.$$

This demonstrates that the eigenstates of the new Hamiltonian are still eigenstates of the charge operator written in the old basis.

b) Show that the state

$$|\tilde{0}\rangle \equiv \cos\theta|0\rangle + \sin\theta d_{-k}^\dagger b_k^\dagger |0\rangle$$

is destroyed by both α_k and β_k. This is the vacuum in the new basis.

$$\alpha^\dagger = \cos\theta_k b_k^+ + \sin\theta_k d_{-k}$$

$$\beta^+ = \cos\theta_k d_{-k}^+ - \sin\theta_k b_k$$

$$\alpha^+\alpha - \beta^+\beta = (\cos\theta\, b^+ + \sin\theta\, d)(\cos\theta\, b + \sin\theta\, d^+) - (\cos\theta\, d^+ - \sin\theta\, b)(\cos\theta\, d - \sin\theta\, b^+)$$

$$= b^+b\cos^2\theta - bb^+(\sin^2\theta)$$

$$+ d^+d(-\cos^2\theta) + dd^+\sin^2\theta$$

$$+ b^+d^+(\cos\theta\sin\theta) + d^+b^+\cos\theta\sin\theta$$

$$+ db\sin\theta\cos\theta + bd\sin\theta\cos\theta$$

Using anti-commutation rules

$$= b^+b - \sin^2\theta - d^+d + \sin^2\theta$$

$$= b^+b - d^+d \quad \checkmark$$

(b)

$$\alpha\{\cos\theta|0\rangle + \sin\theta\, d^+b^+|0\rangle\}$$

$$= (\cos\theta\, b + \sin\theta\, d^+)\{\cos\theta|0\rangle + \sin\theta\, d^+b^+|0\rangle\}$$

$$= \sin\theta\cos\theta\, d^+|0\rangle + \sin\theta\cos\theta\, bd^+b^+|0\rangle$$

flips sign

$$= 0$$