1. Consider an electron in an external magnetic field $\boldsymbol{B}$ directed along the z axis and an electric field $\boldsymbol{E}$ in the y direction.

   (a) Choosing the vector potential to lie along the y axis and describe both the electric and magnetic fields, show that the Hamiltonian may be written in the form,

$$H = \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2(x - x_0 - v_0 t)^2 \,,$$

   and find ω, and v_0 in terms of E, B, e, m and c.

   (b) Show that Schrödinger's equation, $i(\partial/\partial t)\Psi = H\Psi$ is satisfied by the form

$$\Psi(x, y, z, t) = e^{-i\epsilon t/\hbar + imv_0 x/\hbar + ik_z z}\phi_n(x - x_0 - v_0 t) \,,$$

   where ϕ_n refers to a harmonic-oscillator wave function characterized by the frequency ω. Find the expectation of the Hamiltonian for a particle in a state described by $n = 0$.

a) $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\Phi, \quad \vec{B} = \nabla \times \vec{A}$

$$\vec{A} = \hat{y}(Bx - cEt)$$

$$H = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2m}\left(p_y - \frac{e}{c}(Bx - cEt)\right)^2$$

Let $p_y = \text{constant} = \frac{eB}{c}x_0$

$$H = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{e^2B^2}{2mc^2}\left(x - x_0 - c\frac{E}{B}t\right)^2$$

$$x_0 = \frac{p_y c}{eB}, \quad v_0 = \frac{cE}{B}$$

$$H = \frac{p_x^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2(x - x_0 - v_0 t)^2$$

time dependent

Let $\Psi = e^{-i\varepsilon t/\hbar + imv_0x/\hbar + ik_z z}\phi(x - x_0 - v_0 t)$

$$i\hbar\frac{\partial\Psi}{\partial t} = \varepsilon\Psi - v_0\left(\frac{\partial_x\phi_n}{\phi_n}\right)\Psi$$

$$H\Psi = \left[\frac{\hbar^2 k_z^2}{2m} + \frac{\hbar^2 m^2 v_0^2}{2m\hbar^2} + \left(n + \tfrac{1}{2}\right)\hbar\omega - \frac{\hbar^2}{2m}\frac{2mv_0}{\hbar}\frac{\partial_x\phi}{\phi}\right]\Psi$$

$$\varepsilon = \left[\frac{\hbar^2 k_z^2}{2m} + mv_0^2/2 + \left(n + \tfrac{1}{2}\right)\hbar\omega\right]$$

2. Consider the coherent state $|\eta\rangle$ defined by,

$$|\eta\rangle = e^{-\eta^*\eta/2}\exp(\eta a^\dagger)|0\rangle$$

(a) Show that the overlap of two states is given by,

$$\langle\eta'|\eta\rangle = e^{-|\eta'|^2/2-|\eta|^2/2+\eta'^*\eta}$$

(b) Show that the normalized coherent state $|\eta\rangle$ may be rewritten in the following form

$$e^{-|\eta|^2/2}e^{\eta a^\dagger}|0\rangle = e^{\eta a^\dagger-\eta^* a}|0\rangle.$$

Hint: You may wish to use the Baker-Campbell-Hausdorff lemma.

(c) Consider bosonic creation and destruction operators, $a^\dagger$ and a. Consider a linear combination,

$$b = \alpha a + \beta a^\dagger$$

What is the constraint on α and β if one is to demand that $[b, b^\dagger] = 1$?

a) $|\eta\rangle = e^{-\eta^*\eta/2}\exp(\eta a^+)|0\rangle$

$$\langle\eta'|\eta\rangle = e^{-\eta'^*\eta'/2}\langle 0|e^{\eta'^* a}|\eta\rangle$$

$$= e^{-\eta'^*\eta'/2}\langle 0|e^{\eta'^*\eta}|\eta\rangle$$

$$= e^{-\eta'^*\eta'/2}\, e^{\eta'^*\eta}\langle 0|e^{\eta a^+}|0\rangle\, e^{-\eta^*\eta/2}$$

$$= e^{-\eta'^*\eta'/2}\, e^{-\eta^*\eta/2}\, e^{-\eta'^*\eta}$$

b) $$e^{\eta a^+ - \eta^* a}|0\rangle = e^{\eta a^+} e^{-\eta^* a} e^{-\frac{1}{2}\eta^*\eta[a,a^+]}|0\rangle$$

$$= e^{-\frac{1}{2}\eta^*\eta}\, e^{\eta a^+}\underbrace{e^{-\eta^* a}}_{=1}|0\rangle$$

$$= e^{-\frac{1}{2}\eta^*\eta}\, e^{\eta a^+}|0\rangle$$

c) $$b = \alpha a + \beta a^+$$

$$b^+ = \alpha^* a^+ + \beta^* a$$

$$[b, b^+] = \alpha\alpha^* - \beta\beta^* = 1$$

OR $\alpha = \cosh\eta\, e^{i\delta}$

$\beta = \sinh\eta\, e^{-i\gamma}$

for any η, δ, γ