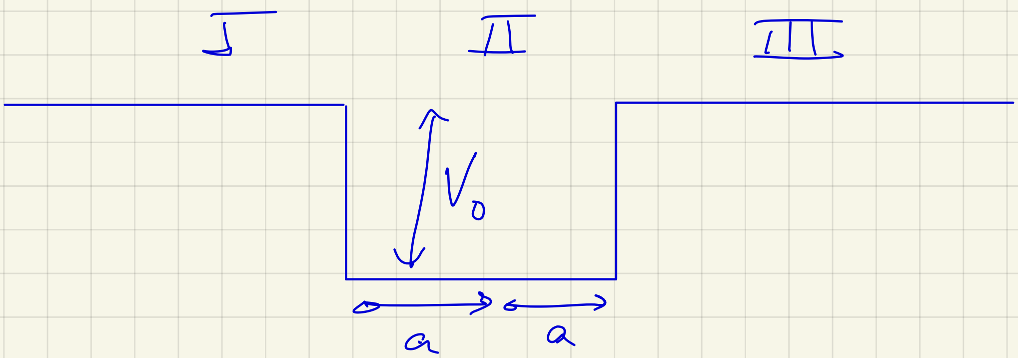
2. Consider the one-dimensional potential,

$$V(x) = \begin{cases} 0, & x < -a \\ -V_0, & -a < x < a \\ 0, & x > a \end{cases}$$

Here, $V_0 > 0$. For fixed a, find the minimum V_0 for the number of bound states to equal or exceed 1,2,3....



For "even" solutions, $\psi_{II} = \cos k_n x$ to match exponential and be barely bound, slope at a & $-a$ must be zero, so

$$k_n a = 0, \pi, 2\pi, 3\pi$$

and because $E = 0$ at threshold,

$$V_0 = \frac{\hbar^2 k_n^2}{2m}, \quad \text{so } V_0 > \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{a^2}, \quad n = 0, 1, 2 \ldots$$

For ground state, $V_0 > 0$

For a 2nd even state, $V_0 > \frac{\hbar^2}{2m} \frac{\pi^2}{a^2}$

For 3rd ..., $V_0 > \frac{\hbar^2}{2m} \frac{4\pi^2}{a^2}$

⋮

For "odd" states, $\psi_{II} = \sin k_n x$

$$k_n a = \pi/2, \ 3\pi/2, \ 5\pi/2 \ \ldots$$

For 1st odd state, $V_0 > \frac{\hbar^2 \pi^2}{2ma^2} \frac{1}{4}$

For 2nd odd state, $V_0 > \frac{\hbar^2}{2m} \frac{\pi^2}{a^2} \frac{9}{4}$

" 3rd ... ... $25/4$

Expressed as formula, for $\geq n$ bound states

$$V_0 > \left(\frac{(n-1)}{2}\right)^2 \frac{\hbar^2 \pi^2}{2ma^2}$$

$$\hookrightarrow V_0 > \frac{\hbar^2}{2ma^2} \begin{cases} 0, & n=1 \\ 1/4, & n=2 \\ 1, & n=3 \\ \frac{9}{4}, & n=4 \\ 4, & n=5 \\ \frac{25}{4}, & n=6 \end{cases}$$

even

odd

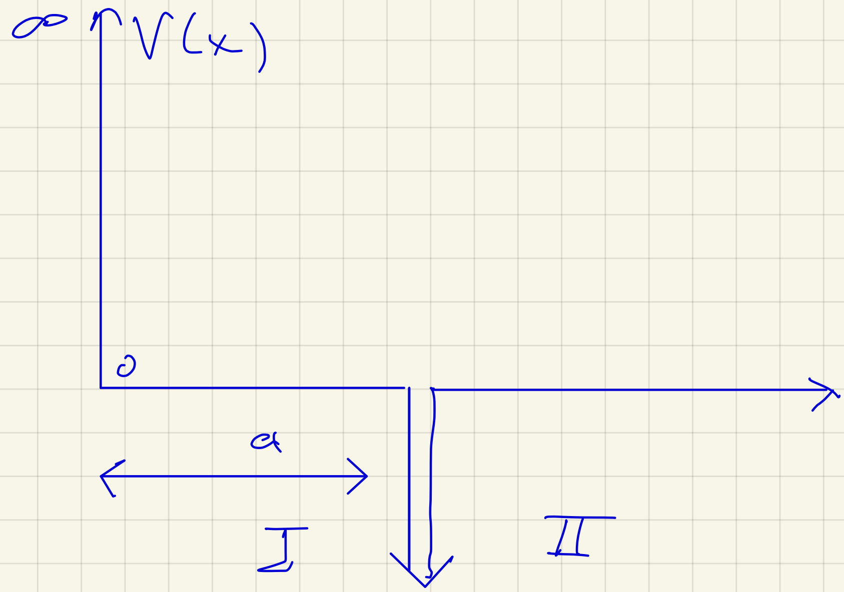
Consider a particle of mass m under the influence of the potential,

$$V(x) = V_0\theta(-x) - \frac{\hbar^2}{2m}\beta\delta(x-a), \quad V_0 \to \infty,\ \beta > 0.$$

(a) Find the transcendental equation for the energy of a bound state?

(b) What is the minimum value of β for a ground state?

(c) For increasing β can one find more than one bound state?



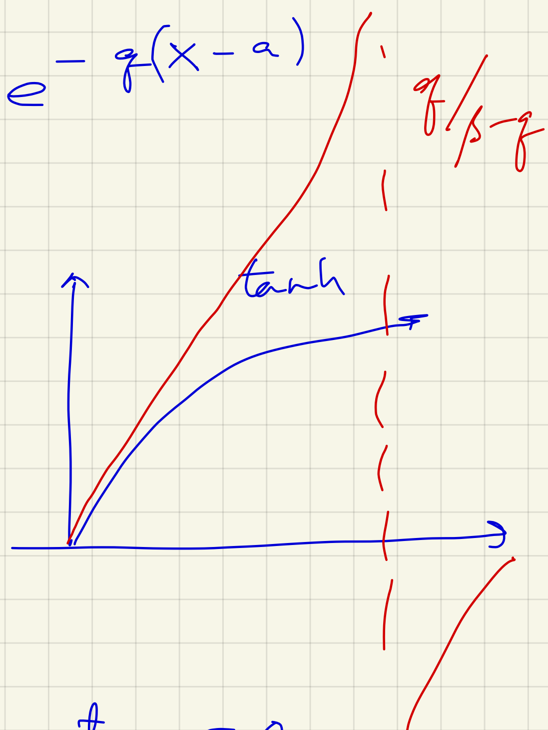
ⓐ $\psi_I = \sinh qx\,, \qquad \psi_{II} = \beta e^{-q(x-a)}$

$\sinh qa = \beta$

$q\cosh qa = -q\beta + \beta\beta$

$\tanh qa = \dfrac{q}{\beta - q}$

$\tanh qa > 0$, so $\beta >$



ⓑ intersects once only, and only if slope is higher at $q=0$ for tanh

$\Rightarrow a > \dfrac{1}{\beta}$ or $\beta > \dfrac{1}{a}$

ⓒ never more than one bound state

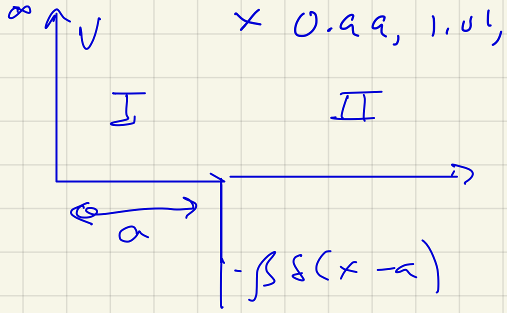
4. Consider plane wave moving in the $-\hat{x}$ direction to be reflected off the potential of the previous problem. For $(x > a)$ the plane wave will have the form

$$e^{-ikx} - e^{2i\delta} e^{ikx}.$$

(a) Find the phase shift δ as a function of ka, and plot for $\beta a = 0.5$.

(b) Repeat for $\beta a = 2$.

$\times$ 0.99, 1.01, 1.5



$$\psi_I = A \sin kx, \quad \psi_{II} = e^{-ikx} - e^{2i\delta} e^{ikx}$$

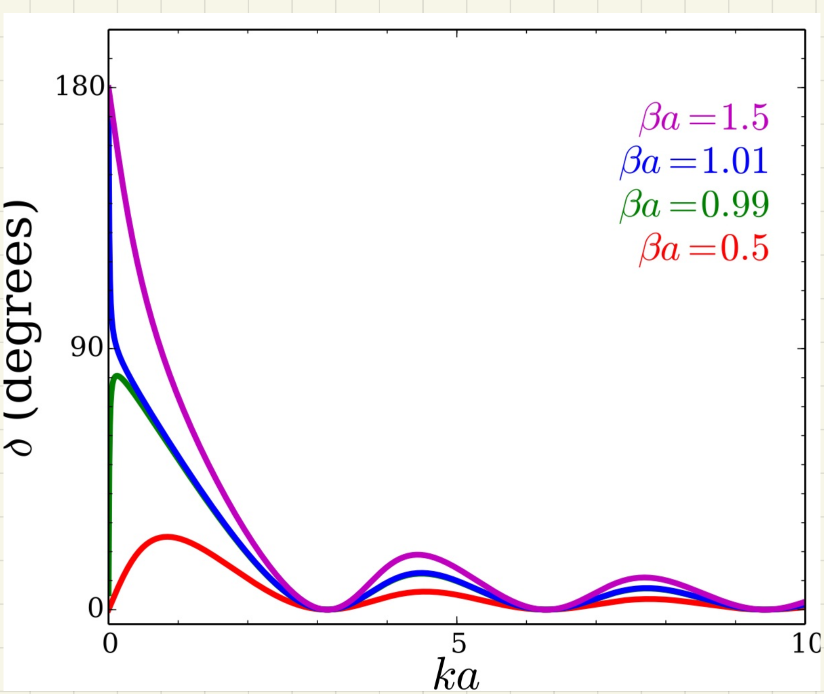
$$A \sin ka = e^{-ika} - e^{2i\delta} e^{ika} = -2ie^{i\delta} \sin(ka+\delta)$$

$$kA \cos ka = -ike^{-ika} - ike^{2i\delta} e^{ika} = -2ike^{i\delta} \cos(ka+\delta) + \beta A \sin ka$$

$$\left(\frac{\sin ka}{k \cos ka - \beta \sin ka}\right) = \frac{-i \sin(ka+\delta)}{-ik \cos(ka+\delta)}$$

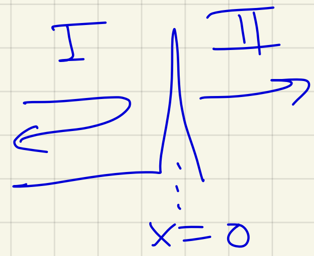
$$\tan(ka+\delta) = \frac{k \sin ka}{k \cos ka - \beta \sin ka} \equiv \alpha$$

$$\delta = -ka + \tan^{-1}(\alpha), \quad \alpha \equiv \frac{k \sin ka}{k \cos ka - \beta \sin ka}$$



(b) Repeat for $\beta a = 2$.

5. Consider a particle of mass m interacting with a repulsive δ function potential,



$$V(x) = \frac{\hbar^2}{2m}\beta\delta(x).$$

Consider particles of energy E incident on the potential.

(a) What fraction of particles are reflected by the potential?

(b) Show that the currents at $x = 0^+$ and $x = 0^-$ are the same.

(a) $\psi_I = e^{ikx} + A e^{-ikx}$, $\psi_{II} = B e^{ikx}$

$1 + A = B$

$ik(1 - A) = ikB - \beta B$

$2ik = 2ikB - \beta B$

$B = \dfrac{1}{1 + i\frac{\beta}{2k}}$, $A = B - 1 = \dfrac{-i\beta/2k}{1 + i\frac{\beta}{2k}}$

fraction reflected $= |A|^2 = \dfrac{\beta^2/4k^2}{1 + \beta^2/4k^2}$

(b)

$J_{left} = k/m - \frac{k}{m}|A|^2 = \frac{k}{m}\dfrac{1}{1 + \beta^2/4k^2}$

$J_{right} = \frac{k}{m}|B|^2 = \frac{k}{m}\dfrac{1}{1 + \beta^2/4k^2}$ ✓

6. Consider a three-dimensional harmonic oscillator with quantum numbers n_x, n_y and n_z. How many states are there with a given $N = n_x + n_y + n_z$? Find a closed expression (no sum). Test it for all $n \leq 3$.

$$M_{xy}(n_{xy}) = \sum_{n_x=0}^{n_{xy}} 1 = (n_{xy}+1)$$

$$M_{xyz}(N) = \sum_{n_{xy}=0}^{N} (n_{xy}+1) = \frac{N(N+1)}{2} + (N+1)$$

$$= \frac{(N+1)(N+2)}{2} = \begin{cases} 1, & N = 0 \\ 3, & N = 1 \\ 6, & N = 2 \\ 10, & N = 3 \end{cases}$$

$N = 0$, $(n_x, n_y, n_z) = (0,0,0)$

$N = 1$, $(n_x, n_y, n_z) = (0,0,1), (0,1,0), (1,0,0)$ 3 ✓

$N = 2$, $(0,0,2)$ $(0,2,0)$, $(2,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ 6 ✓

$N = 3$, $(0,0,3)$, $(0,3,0)$, $(3,0,0)$, $(2,1,0)$, $(2,0,1)$, $(0,2,1)$, $(1,2,0)$, $(0,1,2)$, $(1,0,2)$, $(1,1,1)$ 10 ✓

7. Calculate $\langle 0|aaa^\dagger aa^\dagger a^\dagger|0\rangle$ and $\langle n|a^\dagger a^\dagger a^\dagger a|m\rangle$.

$$\langle 0| a a a^\dagger a a^\dagger a^\dagger a^\dagger |0\rangle = \langle 0| a a N a^\dagger a^\dagger |0\rangle$$
$$= \langle 2| N |2\rangle \cdot 2 = 4$$

$$\langle n| a^\dagger a^\dagger a^\dagger a |m\rangle = \delta_{n,m+2} \langle n| a^\dagger a^\dagger a^\dagger |m-1\rangle \sqrt{m}$$
$$= \delta_{n,m+2} \langle n|m+2\rangle m\sqrt{m+1}\sqrt{m+2}$$

8. Find $\psi_1(x)$, the wave function of the first excited state by applying $a^\dagger$, defined in Eq. (1.55), to the ground state.

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} X - i\sqrt{1/2\hbar m\omega}\, P$$

$$\psi_0 = Z^{-1/2} e^{-x^2/2b^2}, \quad Z = \pi^{1/2} b, \quad b = \sqrt{\hbar/m\omega}$$

$$a^\dagger|0\rangle = \frac{1}{\sqrt{Z}}\left\{\sqrt{\frac{m\omega}{2\hbar}} x - i\sqrt{\frac{1}{2\hbar m\omega}}(-i\hbar)\partial_x\right\} e^{-x^2/2b^2}$$

$$= \frac{1}{\sqrt{Z}}\left\{\sqrt{\frac{m\omega}{2\hbar}} x + \sqrt{\frac{\hbar}{2m\omega}} \frac{x}{b^2}\right\} e^{-x^2/2b^2}$$

$$= \frac{x}{\sqrt{Z}}\left\{\sqrt{\frac{m\omega}{2\hbar}} + \sqrt{\frac{\hbar}{2m\omega}} \frac{m\omega}{\hbar}\right\} e^{-x^2/2b^2}$$

$$= \frac{x}{\sqrt{Z}}\left(\sqrt{\frac{2m\omega}{\hbar}}\right) e^{-x^2/2b^2} = \frac{1}{\sqrt{\pi^{1/2} b}}\sqrt{\frac{2}{b}}\, x e^{-x^2/2b^2}$$

$$= \sqrt{\frac{2}{\pi^{1/2}}}\frac{x}{b} e^{-x^2/2b^2}$$

9. Consider a particle of mass m in a harmonic oscillator with spring constant $k = m\omega^2$.

(a) Write the momentum and position operators for a particle of mass m in a harmonic oscillator characterized by frequency ω in terms of the creation and destruction operators.

(b) Calculate $\langle n|\mathcal{X}^2|n\rangle$ and $\langle n|\mathcal{P}^2|n\rangle$. Compare the product of these two matrix elements to the constraint of the uncertainty relation as a function of n.

(c) Show that the expectation value of the potential energy in an energy eigenstate of the harmonic oscillator equals the expectation value of the kinetic energy in that state.

(a)

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\, X - i\sqrt{\frac{1}{2\hbar m\omega}}\, P$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, X + i\sqrt{\frac{1}{2\hbar\omega\omega}}\, P$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$P = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^+ - a)$$

(b)

$$\langle n|X^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a+a^+)^2|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|aa^+ + a^+a|n\rangle$$

$$= \frac{\hbar}{2m\omega}\cdot(2n+1)$$

$$\langle n|P^2|n\rangle = \frac{\hbar m\omega}{2}(2n+1)$$

$$\langle n|X^2|n\rangle\langle n|P^2|n\rangle = (2n+1)^2\,\frac{\hbar^2}{4}$$

For G.S. $= \hbar^2/4$ ✓

(c)

$$\left\langle \frac{P^2}{2m} \right\rangle = \frac{h\omega}{4}(2n+1)$$

$$\left\langle \frac{1}{2} m\omega^2 X^2 \right\rangle = \frac{\hbar\omega}{4}(2n+1)$$

10. (a) What is the representation of the position operator in the momentum basis – how is $\langle \boldsymbol{p}|\boldsymbol{\mathcal{X}}|\boldsymbol{\Psi}\rangle$ related to $\langle \mathbf{p}|\boldsymbol{\Psi}\rangle$?

(b) Suppose that the potential is $v(\mathrm{x}) = (k/2)x^2$. What is the Schrödinger equation written in momentum space; that is, what is the equation of motion of the amplitude $\langle \boldsymbol{p}|\boldsymbol{\Psi}(t)\rangle$?

(a) $\langle p | \cancel{X} | \psi \rangle = \int_{dx} \langle p | x \rangle \, x \, \langle x | \psi \rangle \, dx$

$= i\hbar \, \partial_p \int \langle p | x \rangle \langle x | \psi \rangle \, dx$

$= i\hbar \, \partial_p \langle p | \psi \rangle$

(b) $H = \dfrac{P^2}{2m} + \dfrac{1}{2} k \cancel{X}^2$

$= \dfrac{1}{2} \hbar^2 k \, \partial_p^2 + \dfrac{P^2}{2m}$

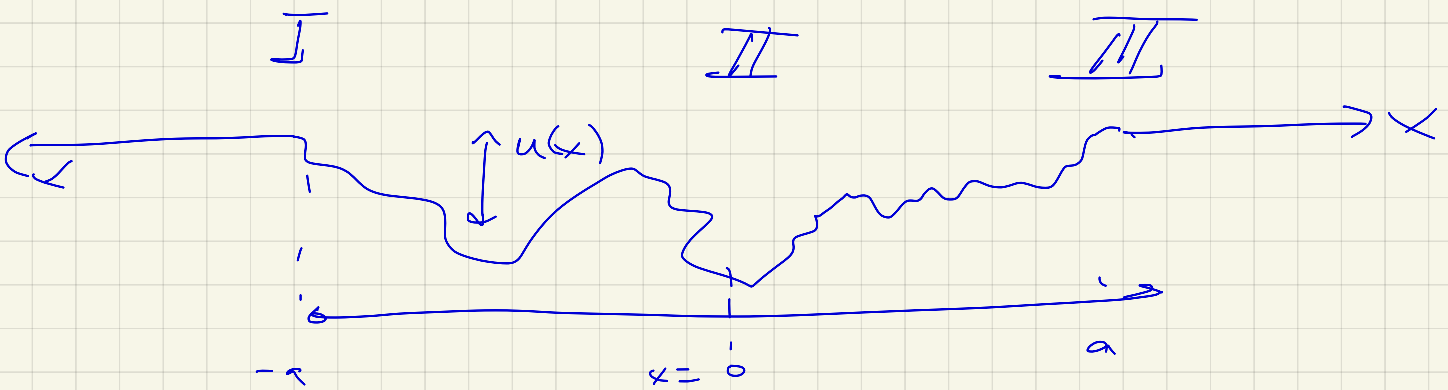
$H \psi(p) = E \psi(p)$

11. Consider a potential

$$V(x) = \begin{array}{ll} 0, & x < -a \\ u(x), & -a < x < a \\ 0, & x > a \end{array}$$

where $u(x)$ is an arbitrary real function. Consider a wave incident from the left. Suppose that the transmission amplitude, defined as the ratio of the transmitted wave at $x = a$ to the incident wave at $x = -a$, is $S(E)$. Now consider a wave incident from the right.

Show that the transmission amplitude, now defined as the ratio of the transmitted wave at $-a$ to the incident wave at a, is also $S(E)$. *(Hint: the Schrödinger equation in this case is a real equation.)*



Consider Solution

$$-\frac{\hbar^2 \partial_x^2}{2m}\psi(x) + u(x)\psi(x) = E\psi(x)$$

with $\psi_I(x<a) = e^{ikx} + Be^{-ikx}$

$\psi_{III}(x>a) = Ce^{ikx}$

Because Sch. eq is real, you can take complex conj. of solution, $\varphi = \psi^*$

$$\varphi_I = e^{-ikx} + B^* e^{ikx}$$

$$\varphi_{III} = C^* e^{-ikx}$$

Now, consider $\chi = B^*\psi - \varphi$

$\chi_I = (B^*B - 1)\, e^{-ikx}$, $\chi_{III} = (B^* C e^{ikx} - C^* e^{-ikx})$

transmission (Right → Left) $= \dfrac{|(B^*B-1)|^2}{|C|^2} = \dfrac{(|C|^2)^2}{|C|^2} = |C|^2$ ✓

12. (a) Derive and solve the equations of motion for the Heisenberg operators $\boldsymbol{a}(t)$ and $\boldsymbol{a}^\dagger(t)$ for the harmonic oscillator.

(b) Calculate $[\boldsymbol{a}(t), \boldsymbol{a}^\dagger(t')]$.

$$\frac{d}{dt} a(t) = \frac{d}{dt} e^{iHt/\hbar} a e^{-iHt/\hbar}$$

$$= \frac{i}{\hbar} e^{iHt/\hbar} [H, a] e^{-iHt/\hbar}$$

$$H = \hbar w \left(a^+ a + \frac{1}{2}\right)$$

$$[H, a] = \hbar w \left(a^+ a a - a a^+ a\right)$$

$$= \hbar w \left(a^+ a a - a^+ a a - a\right)$$

$$= -\hbar w a$$

$$\frac{d}{dt} a(t) = -iw a^t,$$

Similarly

$$\frac{d}{dt} a^+(t) = +iw a^+(t)$$

$$a(t) = e^{-iwt} a$$

$$a^+(t') = e^{iwt'} a^+$$

$$[a(t), a^+(t')] = e^{iw(t'-t)}$$

13. (a) Calculate the correlation function $\langle 0|x(t)x(t')|0\rangle$ where $|0\rangle$ is the harmonic oscillator ground state, and $x(t)$ is the position operator in the Heisenberg representation.

(b) Suppose that a time dependent force $F(t)$ is applied to a particle in the oscillator potential. Show that $x(t)$ obeys the equation of motion,

$$m\left(\frac{d^2}{dt^2}+\omega^2\right)x(t) = F(t)$$

where ω is the oscillator frequency.

(a) From previous problem

$$a(t) = e^{-i\omega t}a, \quad a^+(t) = e^{i\omega t}a^+$$

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}a + e^{i\omega t}a^+\right]$$

$$\langle 0|x(t)x(t')|0\rangle = \frac{\hbar}{2m\omega}\left[\langle 0|\left(e^{-i\omega t}a + e^{i\omega t}\cancel{a^+}\right)\left(e^{-i\omega t'}\cancel{a} + e^{i\omega t'}a^+\right)|0\rangle\right]$$

$$= \frac{\hbar}{2m\omega}e^{i\omega(t'-t)}$$

(b)

$$H = \hbar\omega\left(a^+a+\tfrac{1}{2}\right) - F(t)X$$

$$= \hbar\omega\left(a^+a+\tfrac{1}{2}\right) - \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)F(t)$$

$$\frac{d}{dt}x(t) = \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dt}\left(a(t)+a^+(t)\right)$$

$$= \sqrt{\frac{\hbar\omega}{2m}}\left(-i\,a(t) + i\,a^+(t)\right) - \cancel{\frac{iF(t)}{\hbar}e^{iHt/\hbar}[X,X]e^{-iHt/\hbar}}$$

$$= \frac{p(t)}{m}$$

$$\frac{d}{dt}p(t) = i\sqrt{\frac{\hbar m\omega}{2}}\left(i\omega\,a^+(t) + i\omega\,a(t)\right) - \frac{iF(t)}{\hbar}e^{iHt/\hbar}[X,p]e^{-iHt/\hbar}$$

$$= -m\omega^2 x(t) + F(t).$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$$

$$P = i\sqrt{\frac{\hbar m\omega}{2}}(a^+-a)$$

$$\frac{d^2x}{dt^2} = \omega^2 x(t) + \frac{F(t)}{m}$$

$$m\frac{d^2x(t)}{dt^2} + m\omega^2 x(t) = F(t)$$

14. What are the matrix elements of the operator $\mathbf{1}/|\mathbf{p}|$ in the position representation? That is, find $\langle \mathbf{r}|\mathbf{1}/|\mathbf{p}||\mathbf{r}'\rangle$. Work the problem in three dimensions.

$$\langle \vec{r} | \frac{1}{|\vec{p}|} | \vec{r}\,' \rangle = \int \langle \vec{r} | \vec{q} \rangle \langle \vec{q} | \frac{1}{|\vec{p}|} | \vec{q}\,' \rangle \langle q' | \vec{r}\,' \rangle \frac{d^3q\, d^3q'}{(2\pi)^6}$$

$$= \int e^{+i\vec{q}\,'\cdot\vec{r}\,' - i\vec{q}\cdot\vec{r}} \frac{\delta^3(\vec{q}-\vec{q}\,')}{\hbar q (2\pi)^3} d^3q\, d^3q'$$

$$= \int \frac{d^3q}{(2\pi)^3\hbar} \frac{e^{i\vec{q}\cdot(\vec{r}\,'-\vec{r})}}{|\vec{q}|}$$

$$= \frac{1}{4\pi^2\hbar} \int \frac{q^2 dq\, d\cos\theta}{q} e^{iq|\vec{r}-\vec{r}\,'|\cos\theta}$$

$$= \frac{1}{2\pi^2\hbar} \int_0^\infty dq \frac{\sin(q|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{2\pi^2\hbar} \frac{-1}{|\vec{r}-\vec{r}\,'|^2}$$

15. Calculate the Wigner transform $f(p, x)$ for a particle in the ground state of an infinite square well potential,

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < a \\ \infty, & x > a \end{cases} .$$

Are there any regions with phase space densities either greater than unity or less than zero?

(1) $\psi(x) = \sin(\pi x/a)$, Let $q \equiv \pi/a$

Let $x' \equiv x - a/2$

$\psi(x) = \sqrt{\frac{2}{a}} \cos \frac{\pi x'}{a} = \cos q x'$, $-\frac{a}{2} \le x' < \frac{a}{2}$

Let $x' > 0$

$$f(k,x) = \frac{2}{a} \int^{y_{max}} dy \sin q\left(x + \frac{y}{2}\right) \sin q\left(x - \frac{y}{2}\right) dy\, e^{iky}$$

$$= \frac{2}{a} \int_{-y_{max}}^{y_{max}} dy \cos q\left(x' + \frac{y}{2}\right) \cos q\left(x' - \frac{y}{2}\right) e^{iky}$$

$\leftarrow$ for $x' > 0$

only need $\cos(qy)$ part because it is even

$$f(k,x) = \frac{2}{a} \int_{-y_{max}}^{y_{max}} dy \cos q\left(x' + \frac{y}{2}\right) \cos q\left(x' - \frac{y}{2}\right) \cos ky$$

$y_{max} = a - 2x' = 2(a - x)$

$$f(k,x) = \frac{1}{a} \int_{-y_{max}}^{y_{max}} dy \left[\cos 2qx' + \cos qy\right] \cos ky$$

$$= \frac{2 \cos 2qx' \sin k y_{max}}{ka} + \frac{\sin(q+k) y_{max}}{(q+k)a} + \frac{\sin(q-k) y_{max}}{(q-k)a}$$

$$y_{max} = \begin{cases} 2(a-x), & x > \frac{a}{2} \\ 2x, & x < \frac{a}{2} \end{cases}$$

$x' = x - a/2$