1. Using the equations of motion for the wavefunction, show that the density and current defined by

$$\rho(\vec{r},t) = |\psi(\vec{r},t)|^2,$$

$$\vec{j}(\vec{r},t) = \frac{-i\hbar}{2m}(\psi^*(\vec{r},t)\nabla\psi(\vec{r},t) - (\nabla\psi^*(\vec{r},t))\psi(\vec{r},t)) - \frac{e\vec{A}}{mc}|\psi(\vec{r},t)|^2,$$

satisfies the continuity equation,

$$\partial_t\rho + \nabla\cdot\vec{j} = 0.$$

$$\frac{d\rho}{dt} = \frac{-1}{i\hbar}(H\psi^*)\psi + \frac{1}{i\hbar}\psi^* H\psi$$

$$= \frac{i}{\hbar}\Big\{ \frac{\hbar^2}{2m}(\nabla^2\psi^*)\psi - \frac{\hbar^2}{2m}\psi^*(\nabla^2\psi) - \frac{\hbar e}{m}(\vec{A}\cdot\vec{\nabla}\psi^*)\psi - \frac{\hbar e}{m}\psi^*(\vec{A}\cdot\vec{\nabla}\psi) - \cancel{\psi^*\frac{e^2\vec{A}^2}{2m}\psi} + \cancel{\left(\frac{e^2\vec{A}^2}{2m}\psi^*\right)}\psi \Big\}$$

$$= \vec{\nabla}\cdot\left\{ \frac{(i\nabla\psi^*)\psi - \psi^*(i\nabla\psi)}{2m} - \frac{e\vec{A}}{m}\psi^*\psi \right\}$$

$$= \vec{j}$$

2. Consider a particle of charge e traveling in the electromagnetic potentials

$$\mathbf{A}(\mathbf{r},t) = -\nabla\Lambda(\mathbf{r},t), \qquad \Phi(\mathbf{r},t) = \frac{1}{c}\frac{\partial\Lambda(\mathbf{r},t)}{\partial t}$$

where $\Lambda(\mathbf{r},t)$ is an arbitrary scalar function.

(a) What are the electromagnetic fields described by these potentials?

(b) Show that the wave function of the particle is given by

$$\psi(\mathbf{r},t) = \exp\left[-\frac{ie}{\hbar c}\Lambda(\mathbf{r},t)\right]\psi^0(\mathbf{r},t),$$

where ψ^0 solves the Schrödinger equation with $\Lambda = 0$

(c) Let $V(\mathbf{r},t) = e\Phi(t)$ be a spatially uniform time varying potential. Show that

$$\psi(\mathbf{r},t) = \exp\left[-\frac{ie}{\hbar}\int_{-\infty}^{t}\Phi(t')dt'\right]\psi_0(\mathbf{r},t)$$

is a solution if ψ_0 is a solution with $\Phi = 0$.

(a) $\vec{E} = \frac{1}{c}(\nabla\partial_t\Lambda - \nabla\partial_t\Lambda) = 0$

$\vec{B} = \vec{\nabla}\times(\vec{\nabla}\Lambda) = 0$, $B_i = \varepsilon_{ijk}\partial_j\partial_k\Lambda = 0$

(b) $i\hbar\partial_t\psi = \frac{e}{c}(\partial_t\Lambda)\psi + e^{-\frac{ie}{\hbar c}\Lambda}H_0\psi_0$

$H\psi = \frac{(-i\hbar\nabla - e\vec{A}/c)^2}{2m}e^{-\frac{ie}{\hbar c}\Lambda}\psi_0 + \frac{e}{c}\partial_t\Lambda\psi$

$\vec{A} = -\nabla\Lambda$

$(-i\hbar\nabla - e\vec{A}/c)e^{-\frac{ie}{\hbar c}\Lambda}\psi_0 = e^{-\frac{ie}{\hbar c}\Lambda}\left(-i\hbar\nabla\psi_0 - \frac{e\vec{A}}{c} - \nabla\Lambda\frac{e}{c}\right)\psi_0$

$H\psi = e^{-\frac{ie}{\hbar c}\Lambda}\left(-\frac{\hbar^2}{2m}\nabla^2\psi_0\right) + \frac{e}{c}\partial_t\Lambda\psi$

Comparing 0

$i\hbar\partial_t\psi = H\psi$ ✓

(c) Let $\Lambda(t) = \int_{-\infty}^{t} c\,dt'\,\Phi(t')$, from above,

$\psi = e^{-ie/\hbar c\,\Lambda}\psi_0 = e^{-\frac{ie}{\hbar}\int_{-\infty}^{t}dt'\,\Phi(t')}\psi_0$ ✓

3. For a gauge transformation, described in Eq. (3.7), including the associated the phase change to the wave function ψ, described in Eq. (3.8),

(a) Show that the charge density $e\psi^*\psi$ and the current is unchanged by the gauge transformation

(b) Show that the current

$$\vec{j} = \frac{e}{2m}\left[\psi^*(-i\hbar\boldsymbol{\nabla} - e\vec{A}/c)\psi + (i\hbar\boldsymbol{\nabla}\psi^*)\psi\right]$$

is unchanged.

(c) Show that $\langle\chi|\boldsymbol{H}|\psi\rangle$ is unchanged in a gauge transformation where $\boldsymbol{\Lambda}$ is independent of time.

a) $\psi = e^{-\frac{ie}{\hbar c}\Lambda}\psi_0, \quad \psi^*\psi = \psi_0^*\psi$

↑ pure phase

b) $\frac{e}{2m} e^{\frac{ie}{\hbar c}\Lambda}\psi_0^*(-i\hbar\nabla - 2e\vec{A}/c)\, e^{-\frac{ie}{\hbar c}\Lambda}\psi_0 + \frac{e}{2m}\left(i\hbar\nabla\left(e^{-\frac{ie}{\hbar c}\Lambda}\psi_0^*\right)\right)\psi_0 e^{\cdots}$

$= \frac{e}{2m}\psi_0^*\left(-i\hbar\nabla - \frac{2e\vec{A}}{c}\right)\psi_0 + \frac{e}{2m} i\hbar(\nabla\psi_0^*)\psi_0$
$+ \frac{e}{2m} e^{+\frac{ie}{\hbar c}\Lambda}\psi_0^*(+i\hbar)\left(\frac{ie}{\hbar c}\nabla\Lambda\right)\psi_0 e^{-\frac{ie}{\hbar c}\Lambda}$
$+ \frac{e}{2m}(i\hbar)\left(+\frac{ie}{\hbar c}\right)(\nabla\Lambda)\psi_0^*\psi_0$

Set $\vec{A} = \vec{A}_0 - \nabla\Lambda$

$= \frac{e}{2m}\left(\psi_0^*\left(-i\hbar\nabla - \frac{e\vec{A}_0}{c}\right)\psi_0 + (i\hbar\nabla\psi_0^*)\psi\right)$
$+ 2\frac{e^2}{2mc}(\nabla\Lambda)\psi_0^*\psi - \frac{e^2}{mc}(\nabla\Lambda)\psi^*\psi$

$= \frac{e}{2m}\left(\psi_0^*\left(-i\hbar\nabla - \frac{e\vec{A}_0}{c}\right)\psi_0 + (i\hbar\nabla\psi_0^*)\psi\right)$

c) $\left(-i\hbar\nabla - \frac{e}{c}\vec{A} + \frac{e}{c}\nabla\Lambda\right) e^{-\frac{ie}{\hbar c}\Lambda}\psi_0$

$= e^{\frac{ie}{\hbar c}\Lambda}\left(-i\hbar\nabla - \frac{e}{c}\vec{A} + \frac{e}{c}\nabla\Lambda - i\hbar\left(\frac{-ie}{\hbar c}\nabla\Lambda\right)\right)\cdot\psi_0 e^{-\frac{ie}{\hbar c}\Lambda}$

$= \left(\left(-i\hbar\nabla - \frac{e}{c}\vec{A}\right)\psi_0\right)e^{-\frac{ie}{\hbar c}\Lambda}$

4. Find the function $\Lambda(\vec{r}, t)$ that corresponds to the gauge transformation in Eq. (3.7) responsible for re-expressing the vector potential in Eq. (3.9) to the form of Eq. (3.10), and show that both forms give the same magnetic field.

II. $A_y' = Bx$, $A_z' = A_x' = 0$, $\vec{\nabla} \times \vec{A} = B\hat{z}$

I. $A_\varphi = \frac{1}{2} B\rho$

$A_x = -A_\varphi \sin\varphi$, $A_y = A_\varphi \cos\varphi$, $A_z = 0$

$A_x = -\frac{1}{2} B\rho \frac{y}{\rho} = -\frac{1}{2} By$

$A_y = \frac{1}{2} B\rho \frac{x}{\rho} = \frac{1}{2} Bx$

$\vec{A}' = \vec{A} + \frac{1}{2} Bxy$, $\Lambda = -\frac{1}{2} Bxy$

$(\vec{\nabla} \times \vec{A})_x = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = 0$

$(\vec{\nabla} \times \vec{A})_y = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = 0$

$(\vec{\nabla} \times \vec{A})_z = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = \frac{1}{2} B + \frac{1}{2} B = B$ ✓

5. The expression for the $\bar{v}_y$ in Eq. (3.20) is only valid for non-relativistic velocities, where $|\boldsymbol{E}| << |\boldsymbol{B}|$. For a uniform magnetic field $\boldsymbol{B}\hat{\boldsymbol{z}}$, with no electric field, consider the form for the vector potential in Eq. (3.10). Performing a relativistic boost (Lorentz transformation), but for non-relativistic velocities, in the $\boldsymbol{y}$ direction by a velocity $\boldsymbol{v}_y$, what is the resulting zero$^{\text{the}}$ component of the vector potential $\boldsymbol{A}_0$? Equating this with the electric scalar potential, express the strength of the resulting electric field in terms of $\boldsymbol{v}_y$ and $\boldsymbol{B}$.

In lab frame $A^{(0)} = \Phi = -Ex$, $A^{(y)} = Bx$

Boosted,

$$A^{(0)\prime} = \gamma \Phi + \gamma v A^{(y)}$$
$$= \gamma x (-E + v_y B)$$

Choose $v_y = E/B$ to make electric field disapper.