1. (a) Show that $\vec{r}^2 = x^2 + y^2 + z^2$ commutes with L_z.

$$L_z = (x\,\partial_y - y\,\partial x)\,(-i\hbar)$$

$$[L_z, x^2+y^2+z^2] = x\,\partial_y(x^2+y^2+z^2)(-i\hbar) - y\,\partial_x(x^2+y^2+z^2)(-i\hbar)$$

$$= (2xy - 2yx)(-i\hbar) = 0$$

(b) Show that $\vec{r}\cdot\vec{p}$ commutes with L_z.

$$[L_z, (xP_x + yP_y + zP_z)]$$

$$= (-\hbar^2)\left\{ x\,\partial_y (x\,\partial_x + y\,\partial_y + z\,\partial_z) - y\,\partial_x (x\,\partial_x + y\,\partial_y + z\,\partial_z) \right\}$$

$$+ \hbar^2 \left\{ x\,\partial_x (x\,\partial_y - y\,\partial_x) + y\,\partial_y (x\,\partial_y - y\,\partial_x) + z\,\partial_z (x\,\partial_y - y\,\partial_x) \right\}$$

$$= \hbar^2(-x\,\partial_y + y\,\partial_x + x\,\partial_y - y\,\partial_x) = 0$$

2. Any two rotations, $\vec{\alpha}$ and $\vec{\beta}$, can be written as a single rotation by $\vec{\gamma}$, which in the spin 1/2 basis means

$$e^{i\vec{\beta}\cdot\vec{\sigma}/2}e^{i\vec{\alpha}\cdot\vec{\sigma}/2} = e^{i\vec{\gamma}\cdot\vec{\sigma}/2}$$

Show that the equivalent angle $\vec{\gamma}$ may be written in terms of $\vec{\alpha}$ and $\vec{\beta}$ as

$$\cos(\gamma/2) = \cos(\beta/2)\cos(\alpha/2) - \hat{\beta}\cdot\hat{\alpha}\sin(\alpha/2)\sin(\beta/2)$$
$$\hat{\gamma}\sin(\gamma/2) = \cos(\beta/2)\sin(\alpha/2)\hat{\alpha} + \cos(\alpha/2)\sin(\beta/2)\hat{\beta} + \sin(\beta/2)\sin(\alpha/2)\hat{\alpha}\times\hat{\beta},$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are the corresponding unit vectors. Note that these relations would hold for any rotation, not just the spin 1/2 system.

$$e^{i\vec{\beta}\cdot\vec{\sigma}/2} = \cos\left(\frac{\beta}{2}\right) + i(\hat{\beta}\cdot\vec{\sigma})\sin(\beta/2)$$

$$e^{i\vec{\alpha}\cdot\vec{\sigma}/2} = \cos\left(\frac{\alpha}{2}\right) + i(\hat{\alpha}\cdot\vec{\sigma})\sin(\beta/2)$$

$$e^{i\vec{\beta}\cdot\vec{\sigma}/2}e^{i\vec{\alpha}\cdot\vec{\sigma}/2} = \cos\frac{\beta}{2}\cos\frac{\alpha}{2} + i(\hat{\beta}\cdot\vec{\sigma})\sin\frac{\beta}{2}\cos\alpha/2 + i(\hat{\alpha}\cdot\vec{\sigma})\sin\alpha/2\cos\beta/2 - (\hat{\beta}\cdot\vec{\sigma})(\hat{\alpha}\cdot\vec{\sigma})\sin\frac{\beta}{2}\sin\alpha/2$$

$$= \cos\gamma/2 + i(\hat{\gamma}\cdot\vec{\sigma})\sin\gamma/2$$

$$\hat{\beta}_i\sigma_i\hat{\alpha}_j\sigma_j = i\varepsilon_{ijk}\sigma_k\beta_i\alpha_j + \hat{\beta}\cdot\hat{\alpha}$$
$$= i(\hat{\beta}\times\hat{\alpha})\cdot\vec{\sigma} + \hat{\beta}\cdot\hat{\alpha}$$

terms without σ

$$\cos\gamma/2 = \cos\frac{\beta}{2}\cos\alpha/2 - \hat{\beta}\cdot\hat{\alpha}\sin\beta/2\sin\alpha/2$$

$$\sin\frac{\gamma}{2}\hat{\gamma} = \hat{\beta}\sin\beta/2\cos\alpha/2 + \hat{\alpha}\sin\frac{\alpha}{2}\cos\beta/2 - (\hat{\beta}\times\hat{\alpha})\sin\left(\frac{\alpha}{2}\right)\sin(\beta/2)$$

terms linear in σ

3. Consider the matrices,

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These represent the rotation matrices for angular momentum $S = 1$, $S(S+1) = 2$. Note that the eigenvalues of S_z are -1,0,1 as expected.

(a) Explicitly multiply the matrices to show that

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k.$$

For efficiency, just pick one of the three combinations to check.

(b) Explicitly multiply the matrices to show that

$$\sum_i S_i^2 = 2\hbar^2\mathbb{I}.$$

(a)

$$S_x S_y - S_y S_x = \frac{\hbar^2}{2}\left\{\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right\}$$

$$= i\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar S_z$$

(b)

$$S_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$S_z^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sum_i S_i^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 2\hbar^2\,\mathbb{1}$$

4. Express $e^{iL_z\phi/\hbar} X e^{-iL_z\phi/\hbar}$ in terms of X, Y and Z.

$$L_z = -i\hbar\left(x\partial_y - y\partial_x\right) = -i\hbar\,\partial_\varphi$$

$$e^{iL_z\alpha/\hbar} = e^{\alpha\partial_\varphi}$$

$$\underbrace{e^{iL_z\alpha/\hbar}\left(R\sin\theta\cos\varphi\right)}_{\text{Taylor expansion}} e^{-iL_z\alpha}$$

$$= R\sin\theta\cos(\varphi+\alpha)$$

$$= R\sin\theta\cos\varphi\cos\alpha - R\sin\theta\sin\varphi\sin\alpha$$

$$= RX\cos\alpha - RY\sin\alpha$$

5. Consider the six group elements for the symmetry of the equilateral triangle listed in Sec. 4.3. As a six-by-six matrix, write down the coefficient a_{ij}.

1. $\mathcal{R}_1$, The identity
2. $\mathcal{R}_2$, Rotating by 120°
3. $\mathcal{R}_3$, Rotating by 240°
4. $\mathcal{R}_4$, Reflecting about an axis through the center of the triangle in the ~~60~~ 30° direction
5. $\mathcal{R}_5$, Reflecting about an axis through the center of the triangle in the 90° direction
6. $\mathcal{R}_6$, Reflecting about an axis through the center of the triangle in the ~~120~~ 150° direction

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

6. In terms of ℓ, m_1 and m_2 find expressions for:

(a) $\langle \ell m_1 | L_x^2 | \ell m_2 \rangle$

(b) $\langle \ell m_1 | L_x^2 + L_y^2 | \ell m_2 \rangle$

$$L_+ = L_x + i L_y$$

$$L_- = L_x - i L_y$$

$$L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_x^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+ L_- + L_- L_+\right)$$

$$= \frac{1}{4}\left(L_+^2 + L_-^2 + 2L_x^2 + 2L_y^2 \cancel{- i L_x L_y} \cancel{+ i L_y L_x} \cancel{+ i L_x L_y} \cancel{- i L_y L_x}\right)$$

$$= \frac{1}{4}\left(L_+^2 + L_-^2\right) + \frac{1}{2}\left(L^2 - L_z^2\right)$$

$$L_+ |\ell, m_2\rangle = \sqrt{\ell(\ell+1) - m_2^2 - m_2}\, |\ell, m_2+1\rangle$$

$$L_+^2 |\ell, m_2\rangle = \sqrt{\ell(\ell+1) - m_2^2 - m_2}\sqrt{\ell(\ell+1) - (m_2+1)^2 - (m_2+1)}\, |\ell, m_2+2\rangle$$

① $$\langle \ell, m_1 | L_+^2 | \ell m_2\rangle = \left\{\left[\ell(\ell+1) - m_2^2 - m_2\right]\left[\ell(\ell+1) - (m_2+1)^2 - (m_2+1)\right]\right\}^{1/2} \delta_{m_2+2, m_1}$$

② $$\langle \ell, m_1 | L_-^2 | \ell m_2\rangle = \left\{\left[\ell(\ell+1) - m_2^2 + m_2\right]\left[\ell(\ell+1) - (m_2-1)^2 + (m_2-1)\right]\right\}^{1/2} \delta_{m_2-2, m_1}$$

③ $$\langle \ell, m_1 | \frac{1}{2}\left(L^2 - L_z^2\right) | \ell m_2\rangle = \delta_{m_1, m_2} \frac{1}{2}\left[\ell(\ell+1) - m_1^2\right]$$

$$\text{Answer} = (① + ② + ③) \cdot \hbar^2$$

(b) $$\langle \ell m_1 | L_x^2 + L_y^2 | \ell m_2\rangle$$

$$= \hbar^2\left[\ell(\ell+1) - m_1(m_1+1)\right] \delta_{m_1 m_2}$$

7. (a) In terms of n_x, n_y and n_z, find the energy levels of the three-dimensional harmonic oscillator, by considering the problem in Cartesian coordinates, with the overall wave function factorizing,

$$\Phi(\vec{r}) = \phi_x(x)\phi_y(y)\phi_z(z)$$

Express the answer in terms of $\hbar$, m and ω, and show that the energy is $\hbar\omega(n_x + n_y + n_z + 3/2)$.

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(a) $\Phi = \varphi_x \varphi_y \varphi_z$

Let $-\frac{\hbar^2}{2m}\partial_x^2 \varphi_x + \frac{1}{2}m\omega^2 x^2 \varphi_x = E_x \varphi$

$E_x = (n_x + \frac{1}{2})\hbar\omega$

$\therefore$ ditto for φ_y & φ_z

then

$$\left(-\frac{\hbar^2 p^2}{2m} + \frac{1}{2}m\omega^2 r^2\right)\Phi = (E_x + E_y + E_z)\Phi$$

$$E = (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega$$

(b) How many independent solutions are there for $N = n_x + n_y + n_z = 0, 1$ or 2.

(c) Write expressions for all the wave functions $\Phi(\vec{r})$ with $N = 1$.

(d) By inspection of the answer above, write down the wave functions $\Phi(\vec{r})$ for the three first excited states in spherical coordinates, which are proportional to Y_{11}, Y_{10} and Y_{1-1}.

(b) For $N = 0 \Rightarrow (n_x, n_y, n_z) = (0,0,0)$ 1 sol.

$N = 1 \Rightarrow$ " $= (1,0,0), (0,1,0), (0,0,1)$ 3 sol.s

$N = 2 \Rightarrow$ " $= (1,1,0)\ (1,0,1), (0,1,1),$
$(2,0,0), (0,2,0), (0,0,2)$ 6 sol.s

(c) $$\varphi = \frac{1}{b^{5/2}\pi^{3/4}}\begin{cases} x\, e^{-r^2/2b^2} \\ y\, e^{-r^2/2b^2} \\ z\, e^{-r^2/2b^2} \end{cases}$$ $b = \sqrt{\hbar/m\omega}$

(d) Taking linear combs of first 2 forms

$$\varphi = \begin{cases} \dfrac{1}{2^{1/2}} \dfrac{1}{\pi^{3/4} b^{5/2}} (x+iy)\, e^{-r^2/2b^2} \\ \dfrac{1}{2^{1/2}} \dfrac{1}{\pi^{3/4} b^{5/2}} (x-iy)\, e^{-r^2/2b^2} \\ \dfrac{1}{\pi^{3/4} b^{5/2}}\, z\, e^{-r^2/2b^2} \end{cases}$$

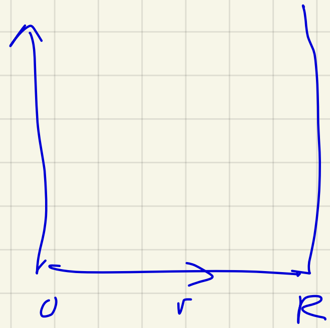
$$= \begin{cases} \dfrac{1}{2^{1/2}} \dfrac{-1}{\pi^{3/4} b^{5/2}} \sqrt{\dfrac{8\pi}{3}}\, Y_{11}\, e^{-r^2/2b^2} \\ \dfrac{1}{2^{1/2}} \dfrac{-1}{\pi^{3/4} b^{5/2}} \sqrt{\dfrac{8\pi}{3}}\, Y_{1-1}\, e^{-r^2/2b^2} \\ \dfrac{1}{\pi^{3/4} b^{5/2}} \sqrt{\dfrac{4\pi}{3}}\, Y_{10}\, e^{-r^2/2b^2} \end{cases}$$

8. Consider a particle of mass m in a spherical well of radius R, where the potential is $+\infty$ for $r > R$ and zero for $r < R$.

(a) Find the ground state energy.

(b) Describe how one would find the energy of the first excited state of the same well.

(c) If the particle is an electron and the radius of the well is 0.15 nm, give a numerical value for the energy of the ground state in eV.



a) For $l = 0$, $u \sim \sin kr$

$$kR = n\pi, \quad k_0 = \pi/R$$

$$E = \frac{1}{2m}\hbar^2\pi^2/R^2$$

b) For $l = 1$, $u \sim \frac{\sin kr}{kr} - \cos kr$

$$0 = \frac{\sin kR}{kR} - \cos kR, \quad \underbrace{\tan kR = kR}_{\text{trans. eq.}}$$

$$E_0 = \frac{(197.326)^2}{2 \cdot 0.511}\pi^2 \frac{1}{(1.5 \cdot 10^5 fm)^2} \text{ MeV}$$

$$= 16.7 \text{ eV}$$

9. Find the ground state binding energies of the following atoms in eV.

   a. e, Pb
   b. μ^-, p
   c. e^+e^-
   d. $\bar{p}, Pb$

$$a_0 = \frac{\hbar^2}{\mu e^2} \frac{1}{Z_1 Z_2}$$

$$E = -\frac{e^2}{2a_0}$$

a) $-13.6 \cdot 82 \text{ eV} = 1.12 \text{ keV}$

b) $$\mu = \frac{m_\mu m_p}{(m_\mu + m_p)} = \frac{205 \cdot 938}{1138} = 169 \text{ MeV}$$

$$= 331 \, m_e$$

$$E_0 = -331 \cdot 13.6 = -4.50 \text{ keV}$$

c) $$\mu = \frac{1}{2} m_e$$

$$E_0 = 27.2 \, m_e$$

d) $$\mu = m_e \frac{938}{0.511} = 1835$$

$$E_0 = -1835 \cdot 13.6 \cdot 82$$
$$= 3.55 \text{ MeV}$$

note wave function would be inside lead nucleus

10. For the same cases above, find the associated Bohr radii.

a. e, Pb

b. μ^-, p

c. e^+e^-

d. $\bar{p}, Pb$

$$a = \frac{\hbar^2}{\mu e^2} \cdot \frac{1}{Z_1 Z_2}$$

$$a_0 = 5.3 \cdot 10^{-11} \text{ m}$$

a) e, Pb

$$a = \frac{1}{82} a_0 = 6.45 \cdot 10^{-13} \text{ m}$$

b) μ^-, P_b , $\mu = m_e \cdot \frac{m_p m_\mu / m_e}{m_\mu + m_p}$

$$= m_e \cdot 329$$

$$a = a_0 / 329 = 1.61 \cdot 10^{-13} \text{ m}$$

c) $a = 1.06 \cdot 10^{-10} \text{ m}$

d) $a = a_0 \frac{1}{82} \cdot \left(\frac{m_e}{m_p} \right)$

$$= 3.52 \cdot 10^{-16} \text{ m}$$

11. For the Hydrogen atom, calculate the expectation of the operator $\boldsymbol{X}$ between the ground state and each of the four $\boldsymbol{n} = \mathbf{2}$ states.

$$X = \frac{r e^{i\phi}}{2} + \frac{r e^{-i\phi}}{2},$$

By symmetry

$$\langle n=1 | X | n=2, \ell=0 \rangle = 0$$

$$\langle n=1 | X | n=2, \ell=1, m=0 \rangle = 0$$

$$\langle n=1 | r e^{-i\phi} | n=2, \ell=1, m=1 \rangle$$

$$= \int r^3 dr \, R_{n=1}(r) \, R_{n=2,\ell=1}(r)$$

From Eq (4-72)

$$= \frac{2}{a_0^{3/2}} \frac{1}{(2a_0)^{3/2}} \frac{1}{a_0\sqrt{3}} \int r^4 dr \, e^{-r/a_0}$$

$$= \frac{a_0}{\sqrt{6}} \cdot 4!$$

$$\langle n=1 | X | n=2, \ell=\pm 1 \rangle$$

$$= \frac{12 a_0}{\sqrt{6}}$$

12. Prove the following recurrence relation for spherical Bessel functions:

$$j_{\ell+1}(z) = -j'_\ell(z) + \frac{\ell}{z} j_\ell(z).$$

Begin with the differential equation for j_ℓ,

$$-j''_\ell(z) - \frac{2}{z} j'_\ell(z) + \frac{\ell(\ell+1)}{z^2} j_\ell(z) = j_\ell(z).$$

Show $j_{\ell+1} = -j_\ell' + \frac{\ell}{z} j_\ell$

$$j'_{\ell+1} = -j_\ell'' - \frac{\ell}{z^2} j_\ell + \frac{\ell}{z} j_\ell'$$

$$= \frac{2}{z} j_\ell' - \frac{\ell(\ell+1)}{z^2} j_\ell + j_\ell - \frac{\ell}{z^2} j_\ell + \frac{\ell}{z} j_\ell'$$

$$= \frac{2+\ell}{z} j_\ell' - \frac{\ell(\ell+2)}{z^2} j_\ell + j_\ell$$

$$j''_{\ell+1} = -\frac{(2+\ell)}{z^2} j_\ell' + \frac{2+\ell}{z} j_\ell'' + \frac{2\ell(\ell+2)}{z^3} j_\ell - \frac{\ell(\ell+2)}{z^2} j_\ell' + j_\ell'$$

$$= \frac{2+\ell}{z} j_\ell'' + \left(-\frac{(\ell+1)(\ell+2)}{z^2} + 1 \right) j_\ell' + \frac{2\ell(\ell+2)}{z^3} j_\ell$$

$$j''_{\ell+1} + \frac{2}{z} j_\ell'$$

$$= \frac{\ell}{z} j_\ell'' + j_\ell' \left(1 - \frac{(\ell+1)(\ell+2)}{z^2} + \frac{2\ell}{z^2} \right) + j_\ell \left(\frac{2\ell(\ell+2)}{z^3} - \frac{2\ell}{z^3} \right)$$

$$= \frac{\ell}{z} j_\ell'' + j_\ell' \left(1 - \frac{\ell^2 + \ell + 2}{z^2} \right) + j_\ell \left(\frac{2\ell(\ell+1)}{z^3} \right)$$

$$\frac{(\ell+1)(\ell+2)}{z^2}\, j_{\ell+1} - j_{\ell+1}$$

$$= \left(\frac{(\ell+1)(\ell+2)}{z^2} - 1\right)\left(-j_\ell' + \frac{\ell}{z}\, j_\ell\right)$$

$$-j_{\ell+1}'' - \frac{2}{z}\, j_{\ell+1}' + \frac{\ell(\ell+1)}{z^2}\, j_\ell - j_\ell \stackrel{?}{=} 0$$

$$= -\frac{\ell}{z}\, j_\ell'' - j_\ell'\left(1 - \frac{\ell^2+\ell+2}{z^2}\right) - j_\ell\left(\frac{2\ell(\ell+1)}{z^3}\right)$$

$$- j_\ell'\left(\frac{(\ell+1)(\ell+2)}{z^2} - 1\right) + j_\ell\left(\frac{\ell(\ell+1)(\ell+2)}{z^3} - \frac{\ell}{z}\right)$$

$$= -\frac{\ell}{z}\, j_\ell'' - j_\ell'\left(\frac{2\ell}{z^2}\right) + j_\ell\left(\frac{\ell^2(\ell+1)}{z^3}\right) - \frac{\ell}{z}\, j_\ell$$

$$= \frac{\ell}{z}\underbrace{\left[-j_\ell'' - \frac{2}{z}\, j_\ell + \frac{\ell(\ell+1)}{z^2}\, j_\ell - j_\ell\right]}_{\text{satisfies diff. eq.}} = 0 \quad ✓$$

13. Find the Clebsch-Gordan coefficient

$$\langle \ell = 1, s = 1, j = 0, m = 0 | \ell = 1, s = 1, m_\ell = 1, m_s = -1 \rangle$$

$$|j=2, m=2\rangle = |m_\ell = 1, m_s = +1\rangle$$

$$J^- |J\ m\rangle = \sqrt{J(J+1) - m^2 + m}$$

$$J^- |j=2, m=2\rangle = (L^- + S^-)|m_\ell = 1, m_s = +1\rangle$$

$$\sqrt{6-4+2}\,|J=2, m=1\rangle = \sqrt{2-1+1}\,|m_\ell = 0, m_s = 1\rangle + \sqrt{2}\,|m_\ell = 1, m_s = 0\rangle$$

$$|J=2, m=1\rangle = \frac{1}{\sqrt{2}}|m_\ell = 0, m_s = 1\rangle + \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = 0\rangle$$

By orthogonality

$$|J=1, m=1\rangle = \frac{1}{\sqrt{2}}|m_\ell = 0, m_s = 1\rangle - \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = 0\rangle$$

$$\sqrt{6}\,|J=2, m=0\rangle = \frac{1}{\sqrt{2}}\left\{\sqrt{2}\,|m_\ell = -1, m_s = 1\rangle + \sqrt{2}\,|m_\ell = 0, m_s = 0\rangle\right\} + \frac{1}{\sqrt{2}}\left\{\sqrt{2}\,|m_\ell = 0, m_s = 0\rangle + \sqrt{2}\,|m_\ell = 1, m_s = -1\rangle\right\}$$

$$|J=2, M=0\rangle = \frac{1}{\sqrt{6}}|m_\ell = -1, m_s = 1\rangle + \frac{1}{\sqrt{6}}|m_\ell = 1, m_s = -1\rangle - \frac{2}{\sqrt{6}}|m_\ell = 0, m_s = 0\rangle$$

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}|m_\ell = -1, m_s = 1\rangle - \frac{1}{\sqrt{2}}|m_\ell = 1, m_s = -1\rangle$$

By orthogonality

$$|J=0, M=0\rangle = \frac{1}{\sqrt{3}}\Big(|m_\ell = 1, m_s = -1\rangle + |m_\ell = -1, m_s = 1\rangle + |m_\ell = 0, m_s = 0\rangle$$

$$\langle J = 0, M = 0 \mid m_\ell = 1, m_s = -1 \rangle$$

$$= \frac{1}{\sqrt{3}}$$

14. Calculate the Clebsch-Gordan Coefficients $\langle \ell = 12, s = 1, j = 12, m_j = 12 | \ell = 12, s = 1, m_\ell, m_s \rangle$ for all m_ℓ and m_s.

$$|J = 13, M_J = 13\rangle = |m_\ell = 12, m_s = 1\rangle$$

$$\sqrt{13 \cdot 14 - 13^2 + 13}\, |J = 13, M_J = 12\rangle = \left(12 \cdot 13 - 12^2 + 12\right)^{1/2} |m_\ell = 11, m_s = 1\rangle + (2)^{1/2} |m_\ell = 12, m_s = 0\rangle$$

$$|J = 13, M_J = 12\rangle = \frac{1}{\sqrt{26}} \left\{ \sqrt{24}\, |m_\ell = 11, m_s = 1\rangle + \sqrt{2}\, |m_\ell = 12, m_s = 0\rangle \right\}$$

$$|J = 12, M_J = 12\rangle = \frac{1}{\sqrt{26}} \left\{ \sqrt{2}\, |m_\ell = 11, m_s = 1\rangle - \sqrt{24}\, |m_\ell = 12, m_s = 0\rangle \right\}$$

$$\langle J = 12, m_J = 12 | \ell = 12, s = 1, m_\ell = 11, m_s = 1\rangle = \frac{1}{\sqrt{13}}$$

$$\langle J = 12, m_J = 12 | \ell = 12, s = 1, m_\ell = 12, m_s = 0\rangle = -\frac{\sqrt{12}}{\sqrt{13}}$$

all other are zero

1. An electron is in an $\ell = 1$ state of a hydrogen atom. It experience a spin orbit interaction,

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$$V_{\text{s.o.}} = \alpha \vec{L} \cdot \vec{S}$$

and also feels an external magnetic field

$$V_b = \mu \vec{B} \cdot \left(\vec{L} + 2\vec{S} \right).$$

(a) Using the basis

$$|m_\ell = 1, m_s = 1/2\rangle = \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \quad |m_\ell = -1, m_s = -1/2\rangle = \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}$$

$$|m_\ell = 1, m_s = -1/2\rangle = \begin{pmatrix} 0\\0\\1\\0\\0\\0 \end{pmatrix}, \quad |m_\ell = 0, m_s = 1/2\rangle = \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix}$$

$$m_\ell = 0, m_s = -1/2\rangle = \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix}, \quad m_\ell = -1, m_s = 1/2\rangle = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}$$

write the Hamiltonian $H = V_{\text{s.o.}} + V_b$ as a 6×6 matrix.

(b) What are the six eigenvalues?

$$V_b = \mu B \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_{s.o.} = \frac{\alpha}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

$$|J = \tfrac{3}{2}, m = \tfrac{3}{2}\rangle = |1, \tfrac{1}{2}\rangle, \quad |J = \tfrac{3}{2}, M = -\tfrac{3}{2}\rangle = |-1, -\tfrac{1}{2}\rangle$$

$$\sqrt{\tfrac{15}{4} - \tfrac{9}{4} + \tfrac{3}{2}}\; |J = \tfrac{3}{2}; M = \tfrac{1}{2}\rangle = \sqrt{2}\, |0, \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{4} - \tfrac{1}{4} + \tfrac{1}{2}}\; |1, -\tfrac{1}{2}\rangle$$

$$|J = \tfrac{3}{2}, m_J = \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0, \tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}} |1, -\tfrac{1}{2}\rangle$$

$$|J = \tfrac{1}{2}, m_J = \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0, \tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |1, -\tfrac{1}{2}\rangle$$

$$|J = \tfrac{3}{2}, m_J = -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0, -\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}} |-1, \tfrac{1}{2}\rangle$$

$$|J = \tfrac{1}{2}, m_J = -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0, -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |-1, \tfrac{1}{2}\rangle$$

$$|J=\tfrac{3}{2}, m_J=\tfrac{3}{2}\rangle = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$$

$$|J=\tfrac{3}{2}, m_J=-\tfrac{3}{2}\rangle = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$$

$$|J=\tfrac{3}{2}, m_J=\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$$

$$|J=\tfrac{1}{2}, M_J=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} - \sqrt{\tfrac{2}{3}}\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$$

$$|J=\tfrac{3}{2}, m_J=-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$|J=\tfrac{1}{2}, m_J=-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} - \sqrt{\tfrac{2}{3}}\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -\sqrt{\frac{2}{3}} \left| J = \frac{1}{2}, M_J = \frac{1}{2} \right\rangle + \frac{1}{\sqrt{3}} \left| J = \frac{3}{2}, M_J = \frac{1}{2} \right\rangle$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \left| J = \frac{1}{2}, M_J = \frac{1}{2} \right\rangle + \sqrt{\frac{2}{3}} \left| J = \frac{3}{2}, M_J = \frac{1}{2} \right\rangle$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2/3} \left| J = \frac{3}{2}, M_J = -\frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}} \left| J = \frac{1}{2}, M_J = -\frac{1}{2} \right\rangle$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\sqrt{\frac{2}{3}} \left| J = \frac{3}{2}, M_J = -\frac{1}{2} \right\rangle + \frac{1}{\sqrt{3}} \left| J = \frac{1}{2}, M_J = -\frac{1}{2} \right\rangle$$

$$H_{so} = \alpha \begin{pmatrix} \frac{1}{2} & & & & & \\ & \frac{1}{2} & & & & \\ & & \left(-\frac{2}{3} + \frac{1}{6}\right) & \left(\frac{\sqrt{2}}{3} + \frac{1}{2}\frac{\sqrt{2}}{3}\right) & & \\ & & '' & \left(-\frac{1}{2} + \frac{1}{3}\right) & & \\ & & & & \left(-\frac{1}{3} + \frac{1}{3}\right) & \left(-\frac{1}{3} - \frac{1}{3}\right) \\ & & & & '' & \left(-\frac{2}{3} + \frac{1}{6}\right) \end{pmatrix}$$

$$\frac{\alpha}{2}\left(\frac{3}{2}\,\frac{5}{2} - 2 - \frac{3}{4}\right) = \frac{\alpha}{2}$$

$$\frac{\alpha}{2}\left(\frac{1}{2}\,\frac{3}{2} - 2 - \frac{3}{4}\right) = -\alpha$$

$$H_{s.o.} = \alpha \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{1}{2} \end{pmatrix}$$

$$H = \begin{pmatrix} \mu B\hbar + \frac{1}{2}\alpha & 0 & & & & \\ 0 & -\mu B\hbar + \frac{1}{2}\alpha & & & & \\ & & -\frac{1}{2}\alpha & \frac{1}{\sqrt{2}}\alpha & & \\ & & \frac{1}{\sqrt{2}}\alpha & \mu B\hbar & & \\ & & & & -\mu B\hbar & -\frac{2}{3}\alpha \\ & & & & -\frac{2}{3}\alpha & -\frac{1}{2}\alpha \end{pmatrix}$$

$$E_n = \mu B\hbar + \frac{1}{2}\alpha, \; -\mu B\hbar + \frac{1}{2}\alpha,$$

$$-\frac{1}{4}\alpha + \frac{\mu B h}{2} \pm \sqrt{\left(-\frac{1}{4}\alpha + \frac{\mu B h}{2}\right)^2 + \alpha^2/2},$$

$$-\frac{1}{4}\alpha - \frac{\mu B\hbar}{2} \pm \sqrt{\left(-\frac{1}{4}\alpha - \frac{\mu B\hbar}{2}\right) + \frac{4}{9}\alpha^2}$$

A spin $\frac{1}{2}$ particle is in $l=0$, spin-up state

$$\Psi(\vec{r}) = \psi(r)\begin{pmatrix}1\\0\end{pmatrix}$$

In terms of $\psi(r)$ and $\vec{r}$, write matrix element for

$$\langle \vec{r}, m_s | (\vec{\sigma}\cdot\vec{r}) | \Psi \rangle$$

for a) $m_s = \frac{1}{2}$ and b) $m_s = -\frac{1}{2}$

$$|J=\tfrac{1}{2}, M=\tfrac{1}{2}, l=0\rangle = |l=0, s=\tfrac{1}{2}, m_l=0, m_s=\tfrac{1}{2}\rangle$$

$$\langle \vec{r}, m_s | (\vec{\sigma}\cdot\vec{r}) | \Psi \rangle$$

$$= \langle m_s | \vec{\sigma} | \chi_{\frac{1}{2}} \rangle \cdot \langle \vec{r} | \vec{r} | \Psi_{rad} \rangle$$

$$\langle m_s | \sigma_x | +\tfrac{1}{2} \rangle = 1 \quad \text{for } m_s = -\tfrac{1}{2}$$
$$= 0 \quad \text{for } m_s = \tfrac{1}{2}$$

$$\langle m_s | \sigma_y | \tfrac{1}{2} \rangle = i \quad \text{for } m_s = -\tfrac{1}{2}$$
$$= 0 \quad \text{for } m_s = \tfrac{1}{2}$$

$$\langle m_s | \sigma_z | \tfrac{1}{2} \rangle = 1 \quad \text{for } m_s = \tfrac{1}{2}$$
$$= 0 \quad \text{for } m_s = -\tfrac{1}{2}$$

For $m_s = -1/2$

$$\langle \vec{r}, m_s | (\vec{\sigma}\cdot\vec{r}) | \Psi \rangle = \psi(r)(x+iy)$$

For $m_s = \frac{1}{2}$

$$= \psi(r)\, z$$