1. Let $\mathcal{T}_{\vec{d}}$ denote the translation operator (displacement vector $\vec{d}$); $\mathcal{D}(\hat{n}, \phi)$, the rotation operator; and π the parity operator. Which, if any, of the following pairs commute? Why?

   (a) $\mathcal{T}_{\vec{d}}$ and $\mathcal{T}_{\vec{d}'}$ ($\vec{d}$ and $\vec{d}'$ are in different directions.)

   (b) $\mathcal{D}(\hat{n}, \phi)$ and $\mathcal{D}(\hat{n}', \phi')$ ($\hat{n}$ and $\hat{n}'$ are in different directions.)

   (c) $\mathcal{T}_{\vec{d}}$ and Π.

   (d) $\mathcal{D}(\hat{n}, \phi)$ and Π.

a) ∂_i + ∂_j commute

b) L_x & L_y don't commute

c) ∂_x & Π don't commute

d) L_i & Π commute

2. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda \left[\delta^3(\mathrm{x})\mathrm{S}\cdot\mathrm{p} + \mathrm{S}\cdot\mathrm{p}\delta^3(\mathrm{x})\right]$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n,\ell,j,m\rangle$ actually con6ains tiny contrihutions from other eigenstates as follows

$$|n,\ell,j,m\rangle \to |n,\ell,j,m\rangle + \sum_{n',\ell',j',m'} C_{n',\ell',j',m'}|n',\ell',j',m'\rangle$$

On the basis of symmetry considerations *alone*, what can you say about (n',ℓ',j',m') which give rise to non-vanishing contributions?

1. $j' = j$, $m' = m$ $\Rightarrow$ angular mom. conservation

2. if ℓ is even ℓ' is odd
   if ℓ is odd ℓ' is even

Not from symmetry, but because wave functions $\sim r^\ell$, one must have $\ell = 0$, and one must be $\ell = 1$, so

$\ell = 0$, $\ell' = 1$ or $\ell = 1$, $\ell' = 0$

3. Suppose a spinless particle is bound to a fixed center by a potential $\boldsymbol{V}(\boldsymbol{x})$ so asymmetrical that no two energy levels are degenerate. Using time-reversal invariance prove

$$\langle \mathbf{L} \rangle = \mathbf{0}.$$

for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.)

$\langle i | \vec{L} | i \rangle$ = real because L is hermitian

$$= \int d^3r\, \psi_i^*(r)\, \vec{L}\, \psi_i(\vec{r})$$

If there is no degeneracy

$\Theta \psi = e^{i\gamma} \psi$, $\psi^* = e^{i\gamma} \psi$

$$= \int d^3r\, \psi_i^*(r)\, i\hbar \partial_\varphi \psi_i(r)$$

$$= \int d^3r\, \psi_i(r) (-i\hbar \partial_\varphi) \psi_i^*(r)$$ because it is real

$$= \int d^3r\, \cancel{e^{-i\gamma}} \psi^*(r)\, i\hbar \partial_\varphi \psi_i(r) \cancel{e^{i\gamma}}$$

$$= \int d^3r\, \psi_i^*(r) (-i\hbar \partial_\varphi) \psi_i(r)$$

Must be zero!

4. Consider the time-reversal operator for spin-1/2 particles, $\Theta = \boldsymbol{\sigma}_y \boldsymbol{K}$, where $\boldsymbol{K}$ takes the complex conjugate of all quantities to its right. Show that Θ commutes with the rotation operator,

$$\boldsymbol{\mathcal{R}}(\vec{\boldsymbol{\theta}}) = \cos(\boldsymbol{\theta}) + i\vec{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\theta}} \sin(\boldsymbol{\theta}).$$

$$[\Theta, i\sigma_x] = \sigma_y K (i\sigma_x) - (i\sigma_x)\sigma_y K$$

because σ_x is real

$$= -i\sigma_y \sigma_x K - i\sigma_x \sigma_y K$$

$$= 0$$

$$[\Theta, i\sigma_y] = \sigma_y K (i\sigma_y) - (i\sigma_y)\sigma_y K$$

because σ_y is imaginary

$$= i\sigma_y^2 K - i\sigma_y^2 K = 0$$

$$[\Theta, i\sigma_z] = \sigma_y K (i\sigma_z) - (i\sigma_z)\sigma_y K$$

because σ_z is real

$$= -i\sigma_y \sigma_z K - i\sigma_z \sigma_y K$$

$$= 0$$

$$\mathcal{R} = \cos\theta\, \mathbb{1} + i\sigma_x \hat{\theta}_x \sin\theta + i\sigma_y \hat{\theta}_y \sin\theta + i\sigma_z \hat{\theta}_z \sin\theta$$

Θ commutes with $\mathcal{R}$ because it commutes with $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$

5 Consider a particle of mass M confined to a two-dimensional circle of radius R.

(a) Write down the Schrödinger equation for the wave function $\psi(\phi)$, where the potential depends only on ϕ, and radial motion is ignored.

(b) Assuming the potential is periodic,

$$V(\phi + 2\pi/N) = V(\phi), \tag{5.34}$$

where N is an integer. Write the boundary condition relating $\psi(\phi)$ and $\psi(\phi + 2\pi/N)$, where the eigenvalue of the rotation operator, $\mathcal{R}(2\pi/N)$ is $e^{i\gamma}$. What values of γ are allowed?

(c) Assume the potential ,

$$V(\phi) = \beta \sum_{j=1,N} \delta(\phi - 2\pi j/N), \tag{5.35}$$

Assume the wave function has the form,

$$\psi(\phi) = e^{im\phi} + Be^{-im\phi}, \quad 0 < \phi < 2\pi/N,$$

where m is not necessarily an integer. Find a transcendental expression for m in terms of β, M, γ and ~~n~~ N.

(d) What are the energies, and degeneracies, of the eigen-states.

a) $-\dfrac{\hbar^2}{2MR^2}\partial_\varphi^2 \psi + V(\varphi)\psi = E\psi$

b) $e^{iN\gamma} = 1, \quad \gamma = 2j\pi/N, \quad j = \text{integer}$

c) $-\dfrac{\hbar^2}{2mR^2}\left(\partial_\varphi \psi\Big|_{\frac{2\pi}{N}+\varepsilon} - \partial_\varphi \psi\Big|_{\frac{2\pi}{N}-\varepsilon}\right) + \beta\psi\Big|_{2\pi/N} = 0$

$$\psi = e^{im\phi} + Be^{-im\varphi}$$

Let $\alpha \equiv 2\pi/N, \quad \gamma = j\alpha$

$\psi(\alpha) = \psi(0)e^{ij\alpha}$

$$e^{im\alpha} + Be^{-im\alpha} = e^{ij\alpha}(1+B)$$

$-\dfrac{\hbar^2}{2mR^2}\left[\psi'(0^+)e^{ij\alpha} - \psi'(\alpha^-)\right] = -\beta\psi(\alpha)$

$$-im\left[e^{im\alpha} - Be^{-im\alpha} - e^{ij\alpha}(1-B)\right] = p(1+B)e^{ij\alpha}$$

$$p = \frac{2M\beta R^2}{\hbar^2}$$

$$e^{im\alpha} + \beta e^{-im\alpha} = e^{ij\alpha}(1+\beta)$$

$$\frac{\hbar^2}{2mR^2}\left[\psi'(0^+)e^{ij\alpha} - \psi'(\alpha^-)\right] = -\beta\,\psi(\alpha)$$

$$-im\left[e^{im\alpha} - \beta e^{-im\alpha} - e^{ij\alpha}(1-\beta)\right] = e^{ij\alpha}\,p\,(1+\beta)$$

$$p = \frac{2M\beta R^2}{\hbar^2}$$

$$\beta = \frac{e^{im\alpha} - e^{ij\alpha}}{e^{ij\alpha} - e^{-im\alpha}}$$

$$\beta = \frac{\frac{p}{m}e^{ij\alpha} + ie^{im\alpha} - ie^{ij\alpha}}{+ie^{-im\alpha} - ie^{ij\alpha} - \frac{p}{m}e^{ij\alpha}}$$

$$\frac{e^{i(m-j)\alpha} - 1}{1 - e^{-i(m+j)\alpha}} = \frac{\left(\frac{p}{m} - i\right) + ie^{i(m-j)\alpha}}{-\left(\frac{p}{m} + i\right) + ie^{-i(m+j)\alpha}}$$

$$i + \left(-\frac{p}{m} - i\right)e^{i(m-j)\alpha} + ie^{-2ij\alpha} - ie^{i(m+j)\alpha}$$

$$= -i + ie^{i(m+j)\alpha} - ie^{-2ij\alpha} - \left(\frac{p}{m} - i\right)e^{-i(m+j)\alpha}$$

$$2i - 2ie^{i(m-j)\alpha} + 2ie^{-2ij\alpha} - 2ie^{-i(m+j)\alpha}$$

$$= \frac{p}{m}\left(-e^{-i(m+j)\alpha} + e^{i(m-j)\alpha}\right)$$

$$4i\cos j\alpha - 4i\cos m\alpha = 2i\frac{p}{m}\sin m\alpha$$

$$2\cos j\alpha - 2\cos m\alpha = \frac{p}{m}\sin m\alpha$$

$$B = \frac{e^{iqa} - e^{ika}}{e^{ika} - e^{-iqa}}$$

$$= \frac{ie^{iqa} - ie^{ika} + \frac{p}{q} e^{ika}}{-\frac{p}{q} e^{ika} + ie^{-iqa} - ie^{ika}}$$

$$-\frac{p}{q} e^{i(k+q)a} + i - ie^{i(k+q)a} + \cancel{\frac{p}{q} e^{2ika}}$$
$$- ie^{i(k-q)a} + ie^{2ika}$$
$$= ie^{i(k+q)a} - ie^{2ika} + \cancel{\frac{p}{q} e^{2ika}}$$
$$- i + ie^{i(k-q)a} - \frac{p}{q} e^{i(k-q)a}$$

$$2i - \frac{p}{q} e^{i(k+q)a} + \frac{p}{q} e^{i(k-q)a} - 2ie^{2i(k-q)a} = 0$$
$$-2ie^{i(k+q)a} + 2ie^{2ika}$$

$$4i\cos ka - 2i\frac{p}{q}\sin qa - 4i\cos qa = 0$$

$$2\cos ka - 2\cos qa - \frac{p}{q}\sin qa = 0$$

$$f = \frac{pa}{qa}\sin qa + 2\cos qa - 2\cos ka = 0$$

$$p\sin qa + 2q\cos qa - 2q\cos ka = 0$$