1. Using the WKB approximation, estimate the lifetime of a particle of mass m initially trapped in the ground state of a one-dimensional rectangular well,

$$V(x) = \begin{cases} \infty, & x < 0 \\ 0, & 0 < x < a \\ \frac{\alpha}{x}, & a < x \end{cases} \tag{6.60}$$

Assume the barrier is sufficiently high that the wave function in the well can be approximated as that of an infinite well and that the frequency of tunneling attempts can be thought of as the rate at which a classical particle would impact the barrier at that energy.

$$P \simeq e^{-2\varphi}, \quad \varphi = \int_a^{x_f} \sqrt{2m(V-E)/\hbar^2}\, dx$$

$$\varphi \simeq \sqrt{\frac{2m}{\hbar^2}} \int_a^{\alpha/E} dx \sqrt{\left(\frac{\alpha}{x} - E\right)}$$

$u = \frac{\alpha}{x}$

$du = -\frac{\alpha}{x^2}\, dx$

$dx = -\frac{1}{u^2}\, du\, \frac{1}{\alpha}$

$$= \sqrt{\frac{2m}{\hbar^2}}\, \frac{1}{\alpha} \int_E^{\alpha/a} \frac{du}{u^2} \sqrt{u - E}$$

$$= \sqrt{\frac{2m}{\hbar^2}}\, \frac{1}{\alpha} \int_0^{\left(\frac{\alpha}{a} - E\right)} \frac{dy}{(y+E)^2}\, y^{1/2}$$

$y^{1/2} = z$

$y = z^2$

$dy = 2z\, dz$

$$\to \int_0^{\left(\frac{\alpha}{aE} - 1\right)^{1/2}} \frac{2z^2\, dz}{(z^2+E)^2}$$

$$= \int \left(2\frac{1}{(z^2+E)} - \frac{2E}{(z^2+E)^2}\right) dz$$

$$I(E) = \int \left(\frac{1}{z^2+E}\right) dz$$

$$= \frac{1}{E^{1/2}} \int_0^{\left(\frac{\alpha}{a} - E\right)^{1/2}} \frac{dw}{w^2+1}$$

$w = z/\sqrt{E}$

$dz = \sqrt{E}\, dw$

$$\varphi = \frac{1}{E^{1/2}} \tan^{-1}\left(\frac{\alpha}{a} - E\right)$$

$$\frac{1}{\tau} = \left(\frac{v}{2a}\right) e^{-2\varphi} = \left(\frac{\hbar\pi/am}{2a}\right) e^{-2\varphi} = \frac{\hbar\pi}{2ma^2} e^{-2\varphi}$$

$$\tau = \frac{2ma^2}{\hbar\pi} e^{2\varphi}$$

2. A particle of mass m is initally in the ground state of a one-dimensional harmonic oscillator of frequency ω centered at $x = 0$. Suddenly, at time $t = 0$, the center of the well is moved to $x = a$.

(a) What is the probability of finding the particle in the ground state of the new well?

(b) What is the expectation of the energy $\langle H \rangle$ after the well is shifted?

(c) If the well were shifted slowly instead of suddenly to its new position, what would be the probability of finding the particle in the ground state of the new well?

(a)

$$|\varphi_0\rangle = e^{\ell \frac{d}{dx}} |0\rangle \qquad b^2 = \frac{\hbar}{m\omega_0}$$

$$\frac{d}{dx} = \frac{ip}{\hbar} = \frac{i}{\hbar}\sqrt{\frac{\hbar m\omega}{2}}\,(a^+ - a) = \frac{i}{2^{1/2}b}(a^+ - a)$$

$$e^{(i\ell/2^{1/2}b)(a^+ - a)} = e^{\frac{i\ell}{2^{1/2}b}a^+} e^{-\frac{i\ell}{2^{1/2}b}a} e^{-\frac{i}{2}\frac{\ell^2}{2b^2}[a,a^+]}$$

$$|\varphi_0\rangle = e^{-\ell^2/4b^2} e^{\frac{i\ell a^+}{2^{1/2}b}} |0\rangle$$

$$= e^{-\ell^2/4b^2} \sum_n^\infty \left(\frac{i\ell}{2^{1/2}b}\right)^n \frac{1}{n!}\sqrt{n!}\,|n\rangle$$

$$\langle\varphi_0|n\rangle = e^{-\ell^2/4b^2}\left(\frac{i\ell}{2^{1/2}b}\right)^n \frac{1}{\sqrt{n!}}$$

$$|\langle\varphi_0|n\rangle|^2 = P_{rob}(0\to n) = e^{-\ell^2/2b^2}\left(\frac{\ell^2}{2b^2}\right)^n \frac{1}{n!}$$

(b)

$$\langle H\rangle = \sum_n e^{-\ell^2/2b^2}\left(\frac{\ell^2}{2b^2}\right)^n \frac{1}{n!}\left(n\hbar\omega_0 + \frac{1}{2}\hbar\omega_0\right)$$

$$= \frac{1}{2}\hbar\omega_0 + \hbar\omega_0\frac{\ell^2}{2b^2}$$

$$= \frac{1}{2}\hbar\omega_0 + \frac{1}{2}m\omega_0^2\ell^2$$

(c) 100 %

3. Estimate the ground state energy of the hydrogen atom using a three-dimensional harmonic oscillator ground state wave function as a trial function.

$$\psi = \frac{1}{(\pi b^2)^{3/4}} e^{-r^2/2b^2}$$

$$\langle KE \rangle = \left\langle \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} \right\rangle$$

$$= 3 \int \frac{dx}{(\pi b^2)^{1/2}} e^{-x^2/2b^2} \partial_x^2 e^{-x^2/2b^2} \cdot -\hbar^2/2m$$

$$= -3 \frac{\hbar^2}{2m} \int \frac{dx}{(\pi b^2)^{1/2}} e^{-x^2/2b^2} \left\{ \frac{x^2}{b^4} - \frac{1}{b^2} \right\} e^{-x^2/2b^2}$$

$$= 3 \frac{\hbar^2}{2m} \left\{ \frac{1}{2b^2} \right\} = \frac{3\hbar^2}{4mb^2}$$

$$\langle V \rangle = -\int \frac{4\pi r^2 dr}{(\pi b^2)^{3/2}} e^{-r^2/b^2} \frac{e^2}{r}$$

$$= \frac{-2\pi e^2}{(\pi b^2)^{3/2}} \int du \, e^{-u/b^2}$$

$$= \frac{-2e^2}{\pi^{1/2} b}$$

$$\frac{d}{db}\left(\frac{3\hbar^2}{4m} \frac{1}{b^2} \right) - \frac{2e^2}{\pi^{1/2} b} = 0$$

$$\frac{3\hbar^2}{2m} \frac{1}{b} = \frac{2e^2}{\pi^{1/2}}, \qquad b = \frac{3\hbar^2 \pi^{1/2}}{4me^2}$$

$$E = \frac{3}{4} \frac{\hbar^2}{m} \left(\frac{4me^2}{3\hbar^2 \pi^{1/2}} \right)^2 - \frac{2e^2}{\pi^{1/2}} \frac{4me^2}{3\hbar^2 \pi^{1/2}}$$

$$= \frac{4me^4}{3\hbar^2\pi} - \frac{8me^4}{3\pi\hbar^2} = -\frac{4}{3} \frac{me^4}{\pi\hbar^2}$$

4. Estimate the ground state energy of the three-dimensional harmonic oscillator using the hydrogen atom wave function as the trial wave function.

$$\psi(r) = e^{-r/b}(\pi b^3)^{-1/2}$$

$$\langle KE \rangle = -4\pi \int r^2 dr \frac{1}{\pi b^3} e^{-r/b}\left(-\partial_r^2 - \frac{2}{r}\partial_r\right)\frac{\hbar^2}{2m} e^{-r/b}$$

$$= -2\frac{\hbar^2}{m b^3}\int r^2 dr\, e^{-r/b}\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) e^{-r/b}$$

$$= -\frac{2\hbar^2}{m b^3}\int r^2 dr\, e^{-2r/b}\left[-\frac{1}{b^2} + \frac{2}{rb}\right]$$

$$= \frac{2\hbar^2}{m b^2}\left\{-2\left(\frac{1}{2}\right)^3 + 2\left(\frac{1}{2}\right)^2\right\}$$

$$= \frac{1\,\hbar^2}{2 m b^2}$$

$$\langle V \rangle = 4\pi \int r^2 dr \frac{1}{\pi b^3} e^{-2r/b}\frac{1}{2} k r^2$$

$$= \frac{2k}{b^3}\int r^4 dr\, e^{-2r/b} = \frac{2\cdot 4!}{b^3} k \cdot \left(\frac{b}{2}\right)^5$$

$$= \frac{3}{2} k b^2$$

$$\frac{d}{db}\left\{\frac{1}{2}\frac{\hbar^2}{m b^2} + \frac{3}{2} k b^2\right\} = -\frac{\hbar^2}{m b^3} + 3kb = 0$$

$$b^2 = \left(\frac{\hbar^2}{3mk}\right)^{1/2}$$

$$E = \frac{1}{2}\frac{\hbar^2}{m}\left(\frac{3mk}{\hbar^2}\right)^{1/2} + \frac{3}{2}k\left(\frac{\hbar^2}{3mk}\right)^{1/2}$$

$$= 3^{1/2}\hbar \frac{k^{1/2}}{m^{1/2}} = \sqrt{3}\,\hbar\omega_0$$

5. Consider a particle in an infinitely deep square well of width a.

$$V_0(x) = \begin{cases} \infty, & x < -a/2 \\ 0, & -a/2 < x < a/2 \\ \infty, & x > a/2 \end{cases}$$

A particle feels a perturbative potential, V_1

$$V_1(x) = \beta \sin(\pi x/a)$$

(a) What is the change in the ground state energy in lowest (non-zero) order perturbation theory?

(b) What is the correction to the energy of the first excited state to the same order?

(c) What is the correction to the wave function of the ground state to lowest order? non-zero

a) $\Delta E = 0$, V_1 is odd

b) $\Delta E = 0$ V_1 is odd

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\langle 0 | V_1 | n\rangle \langle n | V_1 | 0\rangle}{E_n}$

only use terms with $n =$ odd

$$\langle 0 | V | n \rangle = \frac{2}{a} \beta \int_{-a/2}^{a/2} \sin\frac{(n+1)\pi x}{a} \cdot \cos\frac{\pi x}{a} \sin\frac{\pi x}{a} \, dx$$

$$\langle 0 | V | n \rangle = \frac{\beta}{a} \int_{-a/2}^{a/2} \sin\frac{(n+1)\pi x}{a} \sin\frac{2\pi x}{a} \, dx$$

$$= \delta_{n,1} \frac{\beta}{2}$$

$$\mathcal{E}_2 - \mathcal{E}_1 = \frac{\hbar^2}{2m}\left(\frac{4\pi^2}{a^2} - \frac{\pi^2}{a^2}\right) = \frac{3\hbar^2\pi^2}{2ma^2}$$

$$\Delta E^{(2)} = -\frac{ma^2\beta^2}{6\hbar^2\pi^2}$$

6. Consider the Hamiltonian:

$$\boldsymbol{H}_0 = \alpha\boldsymbol{\sigma}_z$$

and the perturbation

$$\boldsymbol{V} = \beta\boldsymbol{\sigma}_x$$

(a) What is the correction to the ground state energy to second order in perturbation theory?

(b) What is the correction to the excited state's energy to the same order?

(c) Write down the exact expression for the energy of the first state, and show that it gives the same answer as part $\boldsymbol{a}$ when expanded in powers of β.

$$\mathcal{E}_0 = -\alpha, \quad \mathcal{E}_1 = \alpha$$

$$\langle 0 | V | 1 \rangle = \beta$$

(a)

For G.S, $\Delta E^{(2)} = -\frac{\beta^2}{2\alpha}$

(b)

For 1st exc. state $\Delta E^{(2)} = +\frac{\beta^2}{2\alpha}$

(c) $E = \pm\sqrt{\alpha^2 + \beta^2}$

$$\simeq \pm\left(\alpha + \frac{\beta^2}{2\alpha} + \cdots\right)$$

7. An electron initially in the ground state of a harmonic oscillator potential is placed in a region with uniform electric field.

(a) By finding corrections to the ground state wave function in first order perturbation theory, write down an expression for the electric dipole moment induced in the atom.

(b) An alternative method for calculating the dipole moment is to differentiate the energy with respect to the electric field. Show that this method yields the same expression found in (a) when one uses second order perturbation theory to find the correction to the energy.

(a) $V = -eEx = -eE(a+a^{+})\sqrt{\hbar/2m\omega_0}$

$$|\psi\rangle = |\psi_0\rangle + \sum_n \frac{\langle 0|V|m\rangle}{(\mathcal{E}_m - \mathcal{E}_0)}|\psi_m\rangle$$

only $n=1$ contributes

$$= |\psi_0\rangle + \frac{eE\sqrt{\hbar/2m\omega}}{\hbar\omega}|\psi_1\rangle$$

$$= |\psi_0\rangle + eE\frac{1}{\hbar^{1/2}m^{1/2}\omega_0^{3/2}}|\psi_1\rangle$$

$e\langle\psi|x|\psi\rangle$ = dipole moment

$$= e\sqrt{\frac{\hbar}{2m\omega_0}}\langle\psi|(a+a^{+})|\psi\rangle$$

$$= 2e^2E\sqrt{\frac{\hbar}{2m\omega_0}}\frac{1}{\sqrt{\hbar m\omega_0}}\frac{1}{\omega_0}$$

$$= \frac{2e^2E}{m\omega_0^2}$$

8. Two electrons whose positions are defined by $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the centers of their confining potentials. The confining potentials are then separated by a distance $\vec{R}$.

$$V_0(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} m\omega^2 (r_1^2 + r_2^2).$$

Positive charges $+e$ are fixed at the centers of the potentials. The electromagnetic energy between the two wells is:

$$V = \frac{e^2}{R} + \frac{e^2}{|\vec{R} + \vec{r}_1 - \vec{r}_2|} - \frac{e^2}{|\vec{R} + \vec{r}_1|} - \frac{e^2}{|\vec{R} - \vec{r}_2|}.$$

Here, the repulsive interaction between the two positive ions is described by the first term, and the repulsive interaction between the electrons is described by the second term. The last two terms describe the attractive interaction between the electron and the ion in the other well. The electromagnetic energy between each electron and its confining ion is assumed to be part of the confining potential, and not part of the perturbation.

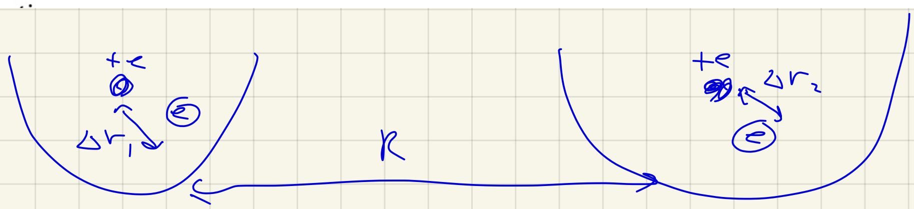
Assume that the separation R is much larger than either r_1 or r_2. In terms of the separation between the wells, R, the mass of the electrons m, the charge e and ω,

(a) Show that for large R, the interaction may be approximated as a dipole-dipole interaction,

$$V = \frac{e^2}{R^3} (x_1 x_2 + y_1 y_2 - 4 z_1 z_2).$$

(b) Use second-order perturbation theory to find the electromagnetic attraction of the two wells, $V(R)$.

a



$$V = e^2/R + \frac{e^2}{|\vec{R} + \Delta\vec{r}_1 - \Delta\vec{r}_2|} - \frac{e^2}{|\vec{R} + \Delta\vec{r}_1|} - \frac{e^2}{|\vec{R} - \Delta\vec{r}_2|}$$

$$|\vec{R} + \vec{\varepsilon}|^2 = (R_x + \varepsilon_x)^2 + (R_y + \varepsilon_y)^2 + (R_z + \varepsilon_z)^2$$

$$= \vec{R}^2 + 2\vec{R}\cdot\vec{\varepsilon} + \vec{\varepsilon}^2$$

$$|\vec{R} + \vec{\varepsilon}| = R\left(1 + \frac{2\vec{\varepsilon}\cdot\vec{R}}{R^2} + \frac{\varepsilon^2}{R^2}\right)^{1/2}$$

$$\cong R\left(1 + \frac{\vec{\varepsilon}\cdot\vec{R}}{R^2} + \frac{\varepsilon^2}{2R^2} - \frac{(\vec{\varepsilon}\cdot\vec{R})^2}{2R^4}\right)$$

$$\frac{1}{|\vec{R} + \vec{\varepsilon}|} \cong \frac{1}{R}\left\{1 - \frac{\vec{\varepsilon}\cdot\vec{R}}{R^2} - \frac{\varepsilon^2}{2R^2} + \frac{(\vec{\varepsilon}\cdot\vec{R})^2}{2R^4} + \frac{(\vec{\varepsilon}\cdot\vec{R})^2}{R^4}\right\}$$

$$= \frac{1}{R}\left\{1 - \frac{\vec{\varepsilon}\cdot\vec{R}}{R^2} - \frac{\varepsilon^2}{2R^2} + \frac{3}{2}\frac{(\vec{\varepsilon}\cdot\vec{R})^2}{R^2}\right\} + O\varepsilon^3$$

$$V = \frac{e^2}{R}\left\{ + \frac{(\Delta r_1)^2}{2R^2} + \frac{\Delta r_2^2}{2R^2} - \frac{(\Delta\vec{r}_1 - \Delta\vec{r}_2)^2}{2R^2} - \frac{3}{2}\left(\frac{\Delta\vec{r}_1\cdot\vec{R}}{R^2}\right)^2 - \frac{3}{2}\left(\frac{\Delta\vec{r}_2\cdot\vec{R}}{R^2}\right)^2 + \frac{3}{2}\left(\frac{(\Delta\vec{r}_1 - \Delta\vec{r}_2)\cdot\vec{R}}{R^2}\right)^2\right\}$$

$$V = \frac{e^2}{R}\left\{ + \frac{\Delta\vec{r}_1 \cdot \Delta\vec{r}_2}{R^2} - \frac{(\Delta\vec{r}_1 \cdot \vec{R})(\Delta\vec{r}_2 \cdot \vec{R})}{R^2} \right\}$$

$$= \frac{e^2}{R^3}\left\{ x_1 x_2 + y_1 y_2 - 2 z_1 z_2 \right\}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

b

$$\langle 0 | V | n_{x_1} = 1, n_{x_2} = 1 \rangle = \frac{\hbar}{2m\omega}\frac{e^2}{R^3}(-2)$$

$$\Delta E^{(2)}_{x} = -\frac{4\frac{\hbar^2}{4m^2\omega^2}\frac{e^4}{R^6}}{2\hbar\omega} = -\frac{\hbar e^4}{8m^2\omega^3}\frac{4}{R^6}$$

$$= \frac{-\hbar e^4}{2m^2\omega^3 R^6}$$

Add contributions from x, y, z to get

$$\Delta E^{(2)}_{TOT} = \frac{-3\hbar e^4}{2m^2\omega^3 R^6}$$

$$= V(R)$$

9. Consider the function

$$g(\omega) \equiv \mathrm{Im}\frac{1}{\omega - i\eta} = \frac{\eta}{\omega^2 + \eta^2}$$

where η is a positive real constant that approaches zero.

(a) What is $g(\omega = 0)$?

(b) What is $g(\omega \neq 0)$?

(c) Using trigonometric substitutions, evaluate

$$\int_{-\infty}^{\infty} d\omega\, g(\omega).$$

(d) Express $g(\omega)$ in terms of a delta function.

a) ∞

b) zero

c) $\int g(w)\,dw = \int \frac{\eta\, dw}{w^2 + \eta^2}$

$\cancel{x} = w/\eta$, $dw = \eta\, dx$

$= \int \frac{dx}{1+x^2}$, $\tan\theta = x$

$dx = (1+\tan^2\theta)\, d\theta$

$1 + x^2 = 1 + \tan^2\theta$

$= \int_{-\pi/2}^{\pi/2} d\theta = \pi$

10. A **bob** particle is in the ground state of a 3-dimensional harmonic oscillator characterized by a frequency ω,

$$V_0 = \frac{1}{2} m\omega^2 r^2$$

A perturbation is added that allows a **bob** particle to undergo a transformation into a **mary** particle. The **mary** particle does not feel the effects of the oscillator potential. The **bob** and **mary** particles have the same mass m. The perturbation is of the form,

$$V_{bm} = g\vec{\epsilon}_s \cdot \int d^3r \psi^*_{bob}(\mathbf{r}) \nabla \psi_{mary}(\mathbf{r}),$$

where ϵ_s is the unit polarization vector of the **mary** particle with polarization s, which refers to any of three directions.

(a) Calculate the lifetime of the **bob** particle.

(b) What is the angular distribution of **mary** particles whose polarization is along the z axis?

(c) The **mary** particle's three polarizations can be expressed in terms of three eigenstates of spin angular momentum $s = 1$. Sketch the angular distributions for **mary** particles that have angular momentum quantum numbers, m_s = -1, 0, or 1.

(a)

$$\psi_m = e^{i\vec{k}\cdot\vec{r}} / \sqrt{V} \qquad k = \sqrt{2mE/\hbar^2}$$

$$E = \frac{1}{2}\hbar\omega$$

$$V_{bm} = g\,\vec{\varepsilon}_s \cdot \int \frac{e^{-r^2/2a^2}}{(\pi a^2)^{3/4}} \vec{\nabla} \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} d^3r$$

$$= \frac{i\vec{k}\cdot\vec{\varepsilon}\, g}{\pi^{3/4} V^{1/2} a^{3/2}} I(k_x) I(k_y) I(k_z)$$

where $I(k_x) = \int dx\, e^{ik_x x} e^{-x^2/2a^2} = (2\pi)^{1/2} a\, e^{-k_x^2 a^2/2}$

$$V_{bm} = \frac{i\vec{k}\cdot\vec{\varepsilon}\, g}{\pi^{3/4} V^{1/2} a^{3/2}} (2\pi)^{3/2} a^3 e^{-k^2 a^2/2}$$

$$= \frac{i\vec{k}\cdot\vec{\varepsilon}\, g}{V^{1/2}} 2^{3/2} \pi^{3/4} a^{3/2} e^{-k^2a^2/2}$$

$$\text{Rate} = \frac{2\pi}{\hbar} \sum_s \int \frac{V d^3k}{(2\pi)^3} |V_{bm}|^2 \delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar\omega}{2}\right) \qquad e^{-k^2a^2}$$

$$\sum_s (\vec{k}\cdot\vec{\varepsilon}_s)^2 = k^2$$

$$\text{Rate} = \frac{2\pi}{\hbar} 4\pi \frac{1}{(2\pi)^3} \int k^2 dk\, k^2 g^2\, 8\pi^{3/2} a^3 e^{-k^2/a^2} \cdot \delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar\omega}{2}\right)$$

$$\delta\left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar\omega}{2}\right) \to \delta\left(k^2 - \frac{m\omega}{\hbar}\right)\frac{2m}{\hbar^2}, \quad k\,dk = \frac{dk^2}{2}$$

$$\text{Rate} = \frac{g^2 \pi^{1/2} \cdot 8 m a^3 k^3}{\hbar^3} e^{-k^2 a^2}$$

(b) $$\frac{dR}{d\Omega} = \frac{2\pi}{\hbar} \frac{1}{(2\pi)^3} \int k^2 dk\, k^2 \cos^2\theta\, g^2 \cdot 8\pi^{3/2} a^3 e^{-k^2/a^2} \delta(\cdots)$$

$$= R \cdot \cos^2\theta / 4\pi$$

total rate from a

(c) $\vec{\varepsilon}(m=0) = \hat{z}$

$\vec{\varepsilon}(m=\pm 1) = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$