7.18 Problems

1. Using the Born approximation estimate the differential scattering cross section, $d\sigma/d\Omega(E)$ for particles of mass m scattering off the following potentials.

    (a) $V(\vec{r}) = V_0\Theta(a-r)$.

    (b) $V(\vec{r}) = a^3V_0\delta^3(\vec{r})$.

    (c) $V(\vec{r}) = a^3V_0\left[\delta^3(\vec{r}-a\hat{z}) + \delta^3(\vec{r}+a\hat{z})\right]$.

    (d) $V(\vec{r}) = a^3V_0\left[\delta^3(\vec{r}-a\hat{z}) - \delta^3(\vec{r}+a\hat{z})\right]$.

    (e) $V(\vec{r}) = a^3V_0\left[\delta^3(\vec{r}-a\hat{x}) - \delta^3(\vec{r}+a\hat{x})\right]$.

    (f) $V_0e^{-r/a}/r$.

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\mathcal{V}(r)e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\right|^2 .$$

First, Calc $\tilde{V}(\vec{q})$

$$= \frac{m^2}{4\pi^2\hbar^4}\left|\tilde{V}(\vec{q})\right|^2$$

a) $$\tilde{V}(\vec{q}) = \int_0^a r^2dr\, d\cos\theta\, d\varphi\, V_0\, e^{iqr\cos\theta}$$

$$= 2\pi V_0\int_0^a r^2dr\,\frac{e^{iqr}-e^{-iqr}}{iqr}$$

$$= 2\pi\frac{V_0}{q}\int_0^a r\,dr\,\sin qr$$

$dv = \sin qr\,dr$

$u = r$

$du = dr$

$v = -\frac{\cos qr}{q}$

$$= \frac{2\pi V_0}{q}\left\{\int_0^a\frac{dr}{q}\cos qr - \frac{a}{q}\cos qa\right\}$$

$$= 2\pi V_0\left\{\frac{\sin qa}{q^3} - \frac{a}{q^2}\cos qa\right\}$$

b) $$\tilde{V}(\vec{q}) = V_0a^3$$

c) $$\tilde{V}(\vec{q}) = V_0a^3\left\{e^{iq_za} + e^{-iq_za}\right\} = 2V_0a^3\cos(q_za)$$

$$= 2V_0a^3\cos\left(ka(1-\cos\theta)\right)$$

d) $$\tilde{V}(\vec{q}) = 2iV_0a^3\sin\left(ka(1-\cos\theta)\right)$$

e) $\tilde{V}(\vec{q}) = -2iV_0 a^3 \sin(ka\cos\theta\cos\varphi)$

f) $\tilde{V}(\vec{q}) = 2\pi V_0 \int_0^\infty r\,dr\, e^{-r/a} \int_{-1}^{1} e^{iqr\cos\theta}\, d\cos\theta$

$$= 2\pi V_0 \int_0^\infty \frac{r\,dr}{qr} \sin qr\, e^{-r/a}$$

$$= \frac{2\pi V_0}{q} \int_0^\infty dr\, \sin qr\, e^{-r/a}$$

$$= \frac{\pi V_0}{iq} \int_0^\infty dr \left(e^{-r/a + iqr} - e^{-r/a - iqr}\right)$$

$$= \frac{\pi V_0}{iq} \left\{ \frac{1}{\frac{1}{a} - iq} - \frac{1}{\frac{1}{a} + iq} \right\}$$

$$= \frac{\pi V_0}{iq} \frac{2iq}{\left(\frac{1}{a}\right)^2 + q^2} = \frac{2\pi V_0}{\left(q^2 + \frac{1}{a^2}\right)}$$

$q = 2k\sin\theta/2,\quad q_z = k(1+\cos\theta)$

plug it into expressions a–f.

ANSWERS IN TERMS OF SCATTERING ANGLES θ, ϕ & $p = \hbar k$

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a) $$\frac{d\sigma}{d\Omega} = \frac{m^2}{\hbar^4} V_0^2 \left\{ \frac{\sin qa}{q^3} - \frac{a}{q^2} \cos qa \right\}^2$$

$$= m^2 \hbar^2 V_0^2 \left[\frac{\sin(2ka\sin\theta/2)}{8(\hbar k)^3 \sin^3(\theta/2)} - \frac{(a/\hbar)}{4\hbar^2 k^2 \sin^2\theta/2} \right]^2$$

b) $$\frac{d\sigma}{d\Omega} = \frac{V_0^2 a^6 m^2}{4\pi^2 \hbar^4}$$

c) $$\frac{d\sigma}{d\Omega} = \frac{V_0 a^6 m^2}{\pi^2 \hbar^4} \cos^2\left[ka(1-\cos\theta)\right]$$

d) $$\frac{d\sigma}{d\Omega} = \frac{V_0 a^6 m^2}{\pi^2 \hbar^4} \sin^2\left[ka(1-\cos\theta)\right]$$

e) $$\frac{d\sigma}{d\Omega} = \frac{V_0 a^6 m^2}{\pi^2 \hbar^4} \sin^2\left[ka \sin\theta \cos\varphi\right]$$

f) $$\frac{d\sigma}{d\Omega} = \frac{V_0^2 m^2}{\hbar^4} \left(\frac{1}{q^2 + (1/a)^2}\right)^2$$

$$= \frac{V_0^2 m^2}{\left(4p^2 \sin^2\frac{\theta}{2} + \left(\frac{1}{a}\right)^2\right)^2}$$

if $\frac{1}{a} \to 0$ gives Rutherford result

2. Show how taking two derivatives of the form factor at $\boldsymbol{q} = \boldsymbol{0}$,

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}F(\vec{q})\bigg|_{q=0},$$

is related to the moments of the charge distribution,

$$\langle r_i r_j \rangle \equiv \int d^3r\, \rho(\vec{r}) r_i r_j.$$

Test your answer by comparing to the result of Example 7.3.

$$F(\vec{q}) = \frac{1}{Ze}\int \rho(\vec{r})\, d^3r\, e^{i\vec{q}\cdot\vec{r}}$$

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j} F(\vec{q}) = \frac{1}{Ze}\int d^3r\, \rho(\vec{r})\,(-r_i r_j)$$

$$= \frac{-1}{Ze}\langle r_i r_j\rangle \cdot \int d^3r\, \rho(\vec{r})$$

$$= -\langle r_i r_j \rangle$$

For Gaussian $F(q) = e^{-q^2a^2/2}$

$$\partial^2/\partial q_x^2\, F(q)\Big|_{q=0} = \frac{\partial}{\partial q_x}\left(a^2 q\, e^{-q^2a^2}\right)\Big|_{q=0}$$

$$= -a^2$$

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j} e^{-q^2a^2/2} = \begin{pmatrix} -a^2 & 0 & 0 \\ 0 & -a^2 & 0 \\ 0 & 0 & -a^2 \end{pmatrix}$$

$$= -\langle r_i r_j \rangle \checkmark$$

3. A π^+, which is a spin-zero meson, scatters off a proton through a Δ^{++} resonance(which is comprised of three up quarks). The Δ^{++} is spin 3/2 baryon. The masses of the pion, proton and delta are 139.58 MeV/c², 938.28 MeV/c² and 1232 MeV/c² respectively. The width of the Δ is 120 MeV.

   (a) Using relativistic dispersion relations, $E = \sqrt{p^2c^2 + m^2}$, what is the relative momentum, q, of the pion and proton at resonance? $\epsilon_\pi(q) + \epsilon_p(q) = M_\Delta$.

   (b) Estimate the cross section at resonance?

(a) $$\sqrt{p^2 + m_\pi^2} + \sqrt{p^2 + m_p^2} = m_\Delta$$

$$\cancel{p^2} + m_\pi^2 = m_\Delta^2 + \cancel{p^2} + m_p^2 - 2m_\Delta\sqrt{p^2 + m_p^2}$$

$$(m_\pi^2 - m_\Delta^2 - m_p^2)^2 = 4m_\Delta^2(p^2 + m_p^2)$$

$$p^2 = \frac{m_\pi^4 + m_\Delta^4 + m_p^4 - 2m_\pi^2 m_\Delta^2 - 2m_\pi^2 m_p^2 - 2m_p^2 m_\Delta^2}{4m_\Delta^2}$$

$$p = 227 \text{ MeV}/c$$

(b) $$\sigma_{MAX} = 2\frac{4\pi\hbar^2}{p^2} = 18.9 \text{ fm}^2$$

$$= 189 \text{ mb}$$

spin factor

4. Consider a particle of mass m that could be confined to a spherical well,

$$V(r) = \begin{cases} 0, & r < a \\ V_0, & a < r < 2a \\ 0, & r > 2a \end{cases}$$

(a) Use the WKB method to estimate the decay rate of a particle of mass m escaping from a spherical trap defined by the potential, Assume the barrier is sufficiently high to approximate the energy of the trapped particle with an infinite well.

(b) Find an expression to estimate the cross section for a particle scattering off the potential well with an energy near the ground state energy described above. Give your answer as a function of the incident energy, E, m, V_0 and a.

(a) $E_R = \dfrac{\hbar^2\pi^2}{2ma^2}$

Prob of tunneling $= \exp -2\int_a^{2a} \dfrac{dx\,\sqrt{2m(V-E_R)}}{\hbar}$

$= \exp\left\{2\left(mV_0 - \dfrac{\hbar^2\pi^2}{2a^2}\right)^{1/2} \cdot \dfrac{a}{\hbar}\right\}$

$\Gamma = \dfrac{\hbar k}{2ma} \cdot P_{\text{tunneling}}$

$= \dfrac{\hbar\pi}{2ma^2}\, P_{\text{tunneling}}$

(b) $\sigma = \dfrac{4\pi}{k^2} \dfrac{(\hbar\Gamma/2)^2}{(\hbar\Gamma/2)^2 + (E-E_R)^2}$

$= \dfrac{4\pi}{k^2} \dfrac{\left(\frac{1}{2}E_R\right)^2 P_{\text{tunnel}}^2}{\left(\frac{1}{2}E_R\right)^2 P_{\text{tunnel}}^2 + (E-E_R)^2}$

$= \dfrac{4a^2}{\pi}\left\{\dfrac{\frac{1}{4}P_{\text{tunnel}}^2 E_R^2}{\frac{1}{4}E_R^2 + (E-E_R)^2}\right\}$

8. One can also show that a second recursion relation is satisfied,

$$\text{a} \quad f_{\ell-1}(x) = \frac{(\ell+1)}{x} f_\ell(x) + \frac{d}{dx} f_\ell(x).$$

Given this recursion relation, plus the one from the previous problem, show that

$$f_{\ell-1}(x) + f_{\ell+1}(x) = \frac{(2\ell+1)}{x} f_\ell(x)$$

Previous Problem

$$\text{c} \quad f_{\ell+1}(x) = \frac{\ell}{x} f_\ell(x) - \frac{d}{dx} f_\ell(x),$$

Adding a + c

$$f_{\ell-1} + f_{\ell+1} = \left(\frac{\ell}{x} + \frac{(\ell+1)}{x} \right) f_\ell = \frac{(2\ell+1)}{x} f_\ell(x)$$

9. Using expressions for j_0, j_1, n_0 and n_1, use recursion relations to find expressions for j_2 and n_2.

$$j_0(x) = \frac{\sin x}{x}, \; n_0(x) = -\frac{\cos x}{x} \tag{7.}$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \; n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$
$$j_2(x) \stackrel{?}{=} \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \; n_2(x) \stackrel{?}{=} -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

$$f_{\ell-1} + f_{\ell+1} = \frac{(2\ell+1)}{x} f_\ell$$

$$j_2 = -j_0 + \frac{3}{x} j_1$$

$$= -\frac{\sin x}{x} - \frac{3}{x^2}\cos x + \frac{3}{x^2}\sin x$$ ✓

$$n_2 = -n_0 + \frac{3}{x} n_1$$

$$= \frac{\cos x}{x} - \frac{3}{x^3}\cos x - \frac{3}{x^2}\sin x$$ ✓

10. Using the recurrence relations show that $j_\ell(z)$ and $n_\ell(z)$ behave as z^ℓ and $z^{-(\ell+1)}$ respectively for $z \to 0$. Begin with the facts that $j_0(z)$ and $n_0(z)$ behave as z^0 and z^{-1} respectively, and that they are even and odd functions in z.

$$j_0 \sim z^0 \quad , \quad n_0 \sim z^{-1}$$

$$f_{\ell+1} = \frac{\ell}{z} f_\ell - \frac{d}{dz} f_\ell$$

FIRST for j_ℓ

If $j_\ell \sim z^\ell$ for some ℓ

$$j_{\ell+1} = \frac{\ell}{z} j_\ell - \frac{d}{dz} j_\ell$$

$$j_\ell = A z^\ell + B z^{\ell+2} + \cdots$$

$$j_{\ell+1} = \cancel{A \ell z^{\ell-1}} - B(\ell+2) z^{\ell+1} + \cdots$$
$$\cancel{- A \ell z^{\ell-1}} + B \ell z^{\ell+1} + \cdots$$

$$= 2B z^{\ell+1} + \cdots$$

thus works for all ℓ

Now FOR n_ℓ

$$n_\ell = A z^{-(\ell+1)} + B z^{-(\ell-1)} + \cdots \quad \text{for some } \ell$$

$$n_{\ell+1} = \frac{\ell}{z} n_\ell - \frac{d}{dz} n_\ell$$

$$= \ell A z^{-(\ell+2)} + (\ell+1) A z^{-(\ell+2)} + \cdots$$

$$= (2\ell+1) A z^{-((\ell+1)+1)} \quad \checkmark$$

thus works for all ℓ

11. (a) Consider a potential which gives non-zero phase shifts for $0 \geq \ell \leq \ell_{max}$, where ℓ_{max} is a large number. Assume these phase shifts can be considered as random numbers, evenly spaced between zero and 2π. Using the expression for the cross section,

$$\sigma = \frac{4\pi\hbar^2}{p^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell,$$

find the overall cross section by averaging over the expectation of the random phases. Give your answer in terms of ℓ_{max} and the incoming momentum p.

(b) Consider a problem classically where one scatters off a strong central potential whose maximum range is R_{max}. From classical arguments, what is the maximum angular momentum of a particle that scatters? Give your answer in terms of R_{max} and the incoming momentum p. What is the total cross section in terms of ℓ_{max} and p?

$$\langle \sin^2\delta \rangle = 1/2$$

$$\sigma = \frac{4\pi}{p^2}\hbar^2 \sum_\ell^{\ell_{max}} (2\ell+1) \cdot \frac{1}{2}$$

$$= \frac{2\pi\hbar^2}{p^2}\left\{(\ell_{max}+1) + 2\ell_{max}(\ell_{max}+1)/2\right\}$$

$$= \frac{2\pi}{p^2}\hbar^2\left\{\ell_{max}^2 + \underline{\mathcal{O}\,\ell_{max}}\right\}$$

$$\hbar\,\ell_{max} = p R_{max}$$

$$\sigma = 2\pi R_{max}^2 \quad \text{(twice the geometric x-section)}$$

$$\sigma_{classical} = \pi R_{max}^2$$

12. Consider a particle of mass m that interacts with a spherically symmetric attractive potential.

$$V(r) = \begin{cases} -V_0, & r < b \\ 0, & r > b \end{cases}$$

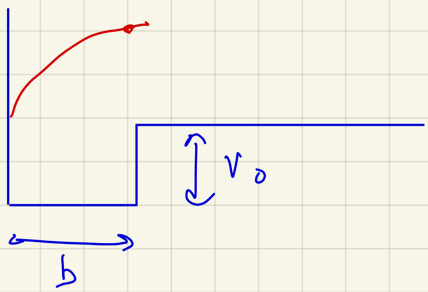
(a) What is the minimum depth $V_{\min}$ that allows a bound state?

(b) Assuming the depth is $V_0 = 0.99 \cdot V_{\min}$, plot the s-wave phase shift for momenta in the range $0 < p < 5\hbar/b$. Use units of $\hbar/b$ for the momenta.

(c) Repeat the above problem for $V_0 = 1.01 \cdot V_{\min}$.

(d) What are the scattering lengths for the two potentials?

(a)



$$q_0 \cdot b = \pi/2$$

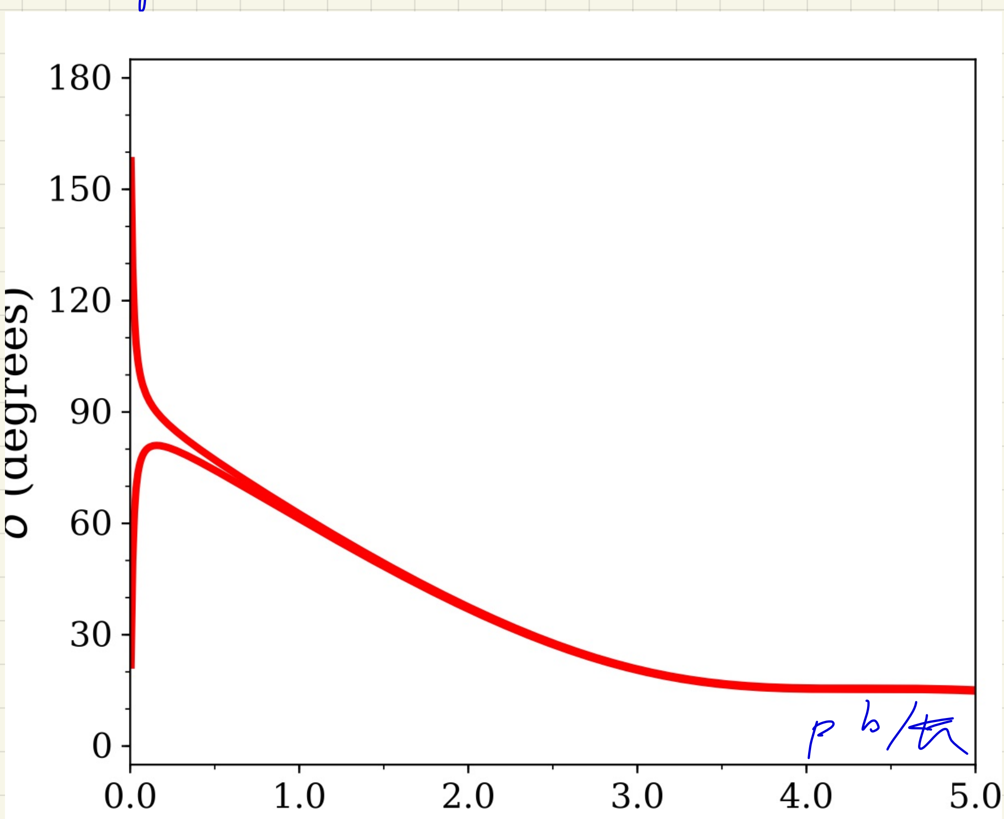
$$\sqrt{2 m V_0}\; b = \pi\hbar/2$$

$$V_0 = \frac{\pi^2 \hbar^2}{8 m b^2}$$

(b, c)

$$\delta = \operatorname{atan}\left[\frac{p}{\hbar q}\tan(q \cdot b)\right] - \frac{p b}{\hbar}$$

$$q = \sqrt{q_0^2 + p^2/\hbar^2}$$



(d) As $p \to 0$

$$\delta = \operatorname{atan}\left[\frac{p}{\hbar q}\tan qb\right] - pb/\hbar$$

$$\Rightarrow \frac{pb}{\hbar}\left\{\frac{\tan qb}{qb} - 1\right\}$$

$$q = \sqrt{2mV_0/\hbar^2}$$

a = scatt. length =

$$b\left\{1 - \frac{\tan qb}{qb}\right\}$$

$$\text{if } q = \frac{\pi}{2b}$$

$$a \to \pm\infty$$

13. Near a resonance of energy ϵ_r, a phase shift behaves as:

$$\tan\delta_\ell = \frac{\Gamma/2}{\epsilon_r - E},$$

where E is the c.m. kinetic energy. For the following problems, assume that $\Gamma << \epsilon_r$, so that the $4\pi/k^2$ prefactor in the expression for the cross section can be considered as a constant.

(a) Write down the cross section $\sigma_\ell(E)$.

(b) What is the maximum cross section (as E is varied) for scattering through that partial wave? (How does it depend on ϵ_r, Γ, the reduced mass μ, and ℓ)?

(c) What is the energy integrated cross section ($\int \sigma_\ell(E)dE$)?

a $\quad \sigma = \frac{4\pi}{k^2}\sin^2\delta = \frac{4\pi}{k^2}\left(1 - \frac{1}{1+\tan^2\delta}\right)$

$= \frac{4\pi}{k_R^2}\,\frac{\tan^2\delta}{1+\tan^2\delta} = \frac{4\pi}{k_R^2}\,\frac{(\Gamma/2)^2}{(E-E_R)^2+\left(\frac{\Gamma}{2}\right)^2}$

b $\quad \sigma_{max} = \frac{4\pi}{k_R^2}$, $\quad k_R = k$ at resonance

c $\quad \int \sigma\, dE \simeq \frac{4\pi}{k_R^2}\int \frac{dx\,\left(\frac{\Gamma}{2}\right)^2}{\left(\frac{\Gamma}{2}\right)^2 + x^2}$

$= \frac{4\pi}{k_R^2}\,\frac{\Gamma}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$, $\quad x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$1 + x^2 = \sec^2\theta$

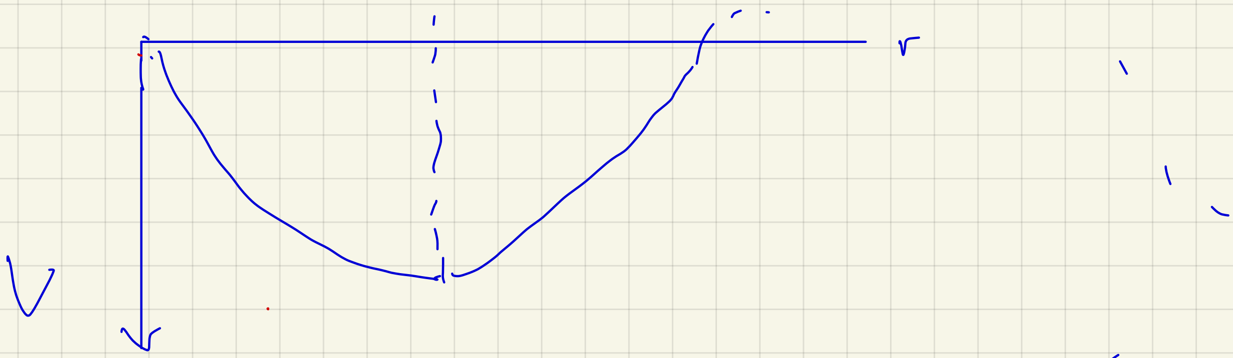
$= \frac{4\pi}{k_R^2}\,\frac{\Gamma}{2}\cdot\pi$

$= \frac{2\pi^2\,\Gamma}{k_R^2}$

14. Consider a particle of mass m interacting with the spherically symmetric attractive potential,

$$V(r) = -\beta\delta(r-b)$$

Find the scattering length as a function of β, b and m.



$$\psi_I = A\sin kr, \quad \psi_{II} = \sin(kr+\delta)$$

$$A\sin(kb) = \sin(kb+\delta)$$

$$kA\cos(kb) = k\cos(kb+\delta) + \frac{2m\beta}{\hbar^2}A\sin(kb)$$

$$\frac{\sin kb}{k\cos kb - \frac{2m\beta}{\hbar^2}\sin kb} = \frac{1}{k}\tan(kb+\delta)$$

$$\delta = -kb + \tan^{-1}\left\{\frac{\sin kb}{\cos kb - \frac{2m\beta}{\hbar^2 k}\sin kb}\right\}$$

$$\left.\frac{d\delta}{dk}\right|_{k=0} = -b + \frac{b}{1 - \frac{2m\beta b}{\hbar^2}}$$

$$= \frac{2m\beta b/\hbar^2}{1-\frac{2m\beta b}{\hbar^2}}\, b = -a$$

15.

15. The temperature at the center of the sun is 15 million degrees Kelvin. Consider two protons with a relative kinetic energy characteristic of the temperature,

$$\frac{\hbar^2 k^2}{2\mu} = \frac{3}{2} kT.$$

(a) What is the Gamow penetrability factor? Give a numeric value.

(b) If the two particles were a proton and a ^{12}C nucleus, what would the penetrability factor become?

(a) $$G = \frac{2\pi\gamma}{e^{2\pi\gamma}-1}, \quad \gamma = \frac{1}{a_0 k},$$

$$kT = 15\cdot 10^6\ K = \frac{15\cdot 10^6\ K}{1.1605\cdot 10^4\ K/eV} = 1.3\ keV$$

$$\frac{\hbar^2 k^2}{2\mu} = \frac{3}{2}\cdot 1.3\ keV$$

$$k = \sqrt{\frac{3\mu}{\hbar}\cdot 1.3\ keV}, \quad \hbar c = 197.326\ eV\ nm$$

$$\hbar c k = 1.91\ MeV$$

$$\gamma = \frac{m_p c^2}{2}\frac{1}{137.036}\frac{1}{(\hbar c)k} = 1.79$$

$$G = 1.44\cdot 10^{-4}, \quad = \frac{2\pi\gamma}{e^{2\pi\gamma}-1}$$

(b) $$\gamma = \frac{m_p\cdot 6}{137.036}\cdot 1.91/\sqrt{2} = 55$$

$$G = 1.25\cdot 10^{-149}$$

$$\gamma \equiv \frac{\mu Z_1 Z_2 e^2}{\hbar^2 k} = -\frac{1}{a_0 k}.$$