1. Consider two oscillator levels described by the creation operators, $a_1^\dagger$ and $a_2^\dagger$, where the Hamiltonian is

$$H = \epsilon_1 a_1^\dagger a_1 + \epsilon_2 a_2^\dagger a_2 + \beta(a_1^\dagger a_2^\dagger + a_1 a_2).$$

Consider the operators

$$b_1^\dagger \equiv \cosh\eta\, a_1^\dagger + \sinh\eta\, a_2,$$
$$b_2^\dagger \equiv \cosh\eta\, a_2^\dagger + \sinh\eta\, a_1.$$

(a) Show that b_i and $b_i^\dagger$ behave like creation/destruction operators.

(b) Find the values of η, E_0, E_1 and E_2 such that allow H to be written as

$$H = E_0 + E_1 b_1^\dagger b_1 + E_2 b_2^\dagger b_2.$$

This is known as a Bogoliubov transformation.

a) $[b_1, b_1^+] = [\cosh\eta\, a_1 + \sinh\eta\, a_2^+, \cosh\eta\, a_1^+ + \sinh\eta\, a_2]$

$= \cosh^2\eta - \sinh^2\eta = 1$ ✓

$[b_1, b_2^+] = [\cosh\eta\, a_1 + \sinh\eta\, a_2^+, \cosh\eta\, a_2^+ + \sinh\eta\, a_1]$

$= \cosh\eta \sinh\eta\,(1-1) = 0$ ✓

take h.c. of previous to show

$[b_2, b_1^+] = 0$ ✓

$[b_2, b_2^+] = [\cosh\eta\, a_2 + \sinh\eta\, a_1^+, \cosh\eta\, a_2^+ + \sinh\eta\, a_1]$

$= \cosh^2\eta - \sinh^2\eta = 1$ ✓

$[b_1^+, b_2^+] = [\cosh\eta\, a_1^+ + \sinh\eta\, a_2, \cosh\eta\, a_2^+ + \sinh\eta\, a_1]$

$= \cosh\eta \sinh\eta - \cosh\eta \sinh\eta = 0$ ✓

Taking h.c. yields $[b_1, b_2] = 0$ ✓

2. Consider heavy b-particles of mass m confined by one-dimensional harmonic oscillator characterized by a frequency ω. The b particles interact with massless and spinless a-particles through their respective field operators,

$$H_{\text{int}} = g \int dx \Psi^\dagger(x)\Phi(x)\Psi(x),$$

where Φ and Ψ are the field operators for the a-particles and b-particles respectively.

$$\Phi(x) = \frac{1}{\sqrt{L}} \sum_k \frac{1}{\sqrt{2E_k}} \left(e^{ikx} a_k^\dagger + e^{-ikx} a_k\right)$$

$$\Psi^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} b_k^\dagger$$

(a) What are the dimensions of g?

(b) What is the decay rate of a b particle in the first excited state.

(a) $[E] = l\left(\frac{1}{\sqrt{l}} \cdot \frac{1}{\sqrt{E}}\right) \cdot \frac{1}{l}[g]$

$[g] = [l^{1/2}][E]^{1/2}[E] = [E]^{3/2}[l]^{1/2}$

(b) $\Gamma = \frac{2\pi}{\hbar} |\langle f|V|i\rangle|^2 \delta(\hbar\omega - \hbar k)$

$\langle n=0, k | V | n=1 \rangle$

$= g \int dx\, \psi_1^*(x) \langle k | \Phi(x) | 0 \rangle \psi_0(x)$

$= g \int dx\, \psi_1^*(x) \psi_0(x) \frac{1}{\sqrt{L}} \frac{1}{\sqrt{E_k}} e^{ikx}$

$e^{ikx} = e^{ik(b+b^\dagger)\sqrt{\frac{\hbar}{2m\omega}}} = e^{ik\sqrt{\frac{\hbar}{2m\omega}} b^\dagger} e^{ik\sqrt{\frac{\hbar}{2m\omega}} b} e^{-\frac{k^2\hbar}{4m\omega}}$

$\langle 1 | e^{ikx} | 0 \rangle = ik\sqrt{\frac{\hbar}{2m\omega}}\, e^{-\hbar k^2/4m\omega}$

$\langle n=0, k | V | n=1 \rangle = \frac{g}{\sqrt{L E_k}} ik\sqrt{\frac{\hbar}{2m\omega}}\, e^{-\hbar k^2/4m\omega}$

$\Gamma_k = \frac{g^2}{L E_k} k^2 \frac{\hbar}{2m\omega} e^{-\hbar k^2/2m\omega} \cdot \frac{2\pi}{\hbar} \delta(\hbar\omega - E_k)$

$= \frac{g^2 \pi k^2}{L E_k m\omega} e^{-\hbar k^2/2m\omega} \delta(\hbar\omega - E_k)$

$$\Gamma = \sum_k \frac{g^2 \pi k^2}{L E_k m \omega} e^{-\hbar k^2/2m\omega} \delta(\hbar\omega - E_k)$$

$$= L \int_0^\infty \frac{dk}{\pi} \frac{g^2 \pi k^2}{L E_k m \omega} e^{-\hbar k^2/2m\omega} \delta(\hbar c k - \hbar\omega)$$

$$= \frac{g^2 k^2}{m E_k \hbar c \omega} e^{-\hbar k^2/2m\omega}$$

$$= \frac{g^2}{m \hbar^2 c^3} e^{-\hbar^2 k^2/2m\omega}$$

3. Show that Eq. (9.33) is satisfied by using the electric and magnetic fields defined in Eq. (9.31).

$$\vec{A}(\vec{r},t) = \sqrt{\frac{2\pi\hbar^2c^2}{V}}\sum_{k,s}\vec{\epsilon}_s(\vec{k})\frac{1}{\sqrt{E_k}}\left(e^{i\vec{k}\cdot\vec{r}-iE_kt/\hbar}a_{k,s} + e^{-i\vec{k}\cdot\vec{r}+iE_kt/\hbar}a^\dagger_{k,s}\right)$$

Let

$$\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E} = \frac{\sqrt{2\pi}}{\sqrt{V}}\sum_k \vec{\epsilon}_s(\vec{k})\, E_k^{1/2}\left(a_{k,s}e^{i\vec{k}\cdot\vec{r}} + a^\dagger_{k,s}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\vec{B} = \frac{-i\sqrt{2\pi}}{\sqrt{V}}\sum_k \left(\vec{\varepsilon}_s\times\hat{k}\right)E_k^{1/2}\left(a_{k,s}e^{i\vec{k}\cdot\vec{r}} - a^\dagger_{k,s}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\int\frac{\left(\vec{E}^2+\vec{B}^2\right)}{8\pi}d^3r = 2\pi\sum_{\substack{k k'\\ s s'}}\int\frac{1}{V}d^3r\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\left(E_k\cdot E_{k'}\right)^{1/2}\left(\vec{\varepsilon}_s\cdot\vec{\varepsilon}_{s'} + (\hat{k}\cdot\varepsilon_s)(\hat{k}\cdot\varepsilon_{s'})\right)\left(a^\dagger_k a_{k'} + a^\dagger_{k'}a_k\right)$$

+ rapidly ~~oscillating~~ term

$$\int\frac{d^3r\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}}{V} = \delta_{\vec{k}\vec{k}'}$$

$$\int\frac{\vec{E}^2+\vec{B}^2}{8\pi}d^3r = \sum_k\frac{2\pi}{8\pi}E_k\left(a^\dagger_k a_k + a_k a^\dagger_k\right)\cdot 2$$

using
$\vec{\varepsilon}_s\cdot\vec{\varepsilon}_{s'} = \delta_{ss'}$
$(\hat{k}\times\vec{\varepsilon}_s)(\hat{k}\times\varepsilon_{s'}) = \delta_{ss'}$

$$= \sum_k E_k\left(a^\dagger_k a_k + \frac{1}{2}\right) \checkmark$$

4. A proton in a nucleus decays from an excited state to its ground state by emitting a photon of momentum $\hbar\vec{k}$ and polarization ϵ_s. The matrix element describing the decay is

$$\langle 0, k, s|V|1\rangle = \beta\vec{\epsilon}_s \cdot \int d^3r \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{V}} \left(\phi_0^*(\vec{r})\nabla\phi_1(\vec{r}) - [\nabla\phi_0^*(\vec{r})]\phi_1(\vec{r})\right).$$

The factor β absorbed all the various factors involved in defining the vector field in Eq. (9.31). Assume the ground and excited states are modeled with a three-dimensional harmonic oscillator of frequency ω. If the excited state is in the first level of a harmonic oscillator and has an angular momentum projection m, what is the angular distribution, $d\Gamma/d\Omega$ of the decay for each m. Remember that the two polarizations of the photon must be perpendicular to $\vec{k}$. You need only calculate the angular shape of the distribution – ignore the prefactors.

$$i\hbar\nabla = \sqrt{\frac{\hbar m\omega}{2}}\, i\,(a - a^+)$$

$$\langle n=0|\nabla|n=1\rangle = \sqrt{\frac{m\omega}{2\hbar}} \quad \text{for 1-d oscillator}$$

$$|m=0\rangle = |n_x = n_y = 0, n_z = 1\rangle$$

$$|m=\pm 1\rangle = \frac{1}{\sqrt{2}}|n_x = 1, n_y = n_z = 0\rangle \pm i\frac{1}{\sqrt{2}}|n_x = n_z = 0, n_y = 1\rangle$$

$$\langle n=0, m=0|\partial_i|n=1, m=0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\cdot\delta_{iz}$$

$$\langle n=0, m=0|\partial_i|n=1, m=\pm 1\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{1}{\sqrt{2}}\delta_{ix} \pm i\frac{1}{\sqrt{2}}\delta_{iy}\right)$$

$$\vec{\epsilon}\cdot\langle n=0|\vec{\nabla}|n=1, m=0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,\epsilon_z$$

$$\vec{\epsilon}\cdot\langle n=0|\vec{\nabla}|n=1, m=\pm 1\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,\frac{1}{\sqrt{2}}(\epsilon_x \pm i\epsilon_y)$$

For $m=0, n=1$, decays go as

$$\sum_s (\vec{\epsilon}_s(\vec{k})\cdot\hat{z})^2 = 1 - (\hat{k}\cdot\hat{z})^2 = \sin^2\theta$$

For $m=\pm 1, n=1$, decays go as

$$\frac{1}{2}\sum_s |\vec{\epsilon}_s(\vec{k})\cdot(\hat{x}\pm i\hat{y})|^2 = 1 - \frac{|\hat{k}\cdot(\hat{x}\pm i\hat{y})|^2}{2} = 1 - \left|\frac{k_x \pm ik_y}{k}\right|^2\frac{1}{2}$$

$$= 1 - \frac{1}{2}\sin^2\theta = \frac{1}{2} + \frac{\cos^2\theta}{2}$$

5. A spinless particle of mass m and charge e is in the first excited state of a three-dimensional harmonic oscillator characterized by a frequency ω. Assume the harmonic oscillator in the Cartesian state with $n_z = 1$. Using the interaction

$$H_{\text{int}} = \vec{j} \cdot \vec{A}/c,$$

(a) Calculate the decay rate of the charged particle into the ground state of the oscillator in the dipole approximation.

(b) Calculate $d\Gamma/d\Omega$ as a function of the emission angles of the photon, θ and ϕ.

(c) In terms of the unit vectors $\hat{k}$, $\hat{\theta}$ and $\hat{\phi}$, write the two polarization vectors which are allowed for emission of a photon at an angle θ, ϕ.

(d) For each polarization vector above, calculate $d\Gamma_s/d\Omega$, the probability of decaying via emission of a photon emited in the θ, ϕ direction with polarization s.

$$\Gamma_{k,s} = \frac{2\pi}{\hbar} \left| e \langle f | \frac{\vec{P} \cdot \vec{A}}{mc} | i \rangle \right|^2 \delta(\hbar\omega = \hbar c k)$$

$$\langle \vec{k}, s | \vec{A} | 0 \rangle \simeq \vec{\varepsilon}_s(\vec{k}) \sqrt{\frac{2\pi\hbar^2 c^2}{V}} \frac{1}{\sqrt{E_k}}$$

dipole approx.

$$\langle 0 | P_z | n=1, n_z = 1 \rangle$$

$$= \frac{i m \hbar \omega}{\hbar} \langle 0 | z | n_z = 1 \rangle$$

!

$$= i m \omega \cdot \sqrt{\frac{\hbar}{2m\omega}} = i \sqrt{\frac{\hbar m \omega}{2}}$$

$$\langle 0 | P_x | n=1, n_z = 1 \rangle = \langle 0 | P_y | n=1, n_z = 1 \rangle = 0$$

$$\Gamma_{k,s} = \frac{2\pi e^2 \hbar m \omega \, 2\pi \hbar^2}{\hbar m^2 \, 2 \, V} \frac{1}{E_k} \left(\vec{\varepsilon}_s(\vec{k}) \cdot \hat{z} \right)^2 \delta(\hbar\omega - \hbar c k)$$

$$\sum_{k,s} \Gamma_{k,s} = \frac{2\pi^2 e^2 \hbar}{m} \sum_s \int \frac{k^2 \, dk \, d\Omega}{(2\pi)^3} \left(\vec{\varepsilon}_s(\vec{k}) \cdot \hat{z} \right)^2 \delta(\hbar\omega - \hbar c k)$$

$$= \frac{2\pi^2 e^2}{m c (2\pi)^3} k^2 \int d\Omega \, (1 - \cos^2\theta)$$

(b) $$\frac{d\Gamma}{d\Omega} = \frac{e^2 k^2}{4\pi m c} \sin^2\theta = \frac{e^2 \omega^2}{4\pi m c^3} \sin^2\theta$$

b) $\int \frac{d\Gamma}{d\Omega} d\Omega = \frac{e^2 \omega^2}{4\pi m c^3} \int d\Omega \sin^2\theta$

$$= \frac{e^2 \omega^2}{m c^3} \left(1 - \frac{1}{3}\right) = \frac{2}{3} \frac{e^2 \omega^2}{m c^3}$$

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c) $\vec{\varepsilon}_\varphi = \hat{\varphi}$, $\varepsilon_\theta = \hat{\theta}$

d) $$\frac{d\Gamma_s}{d\Omega} = \frac{e^2 \omega^2}{4\pi m c^3} \left(\vec{\varepsilon}_s \cdot \hat{z}\right)^2$$

$$= \frac{e^2 \omega^2}{4\pi m c^3} \left(\vec{\varepsilon}_s \cdot \hat{z}\right)^2$$

$$\frac{d\Gamma_\varphi}{d\Omega} = 0$$

$$\frac{d\Gamma_\theta}{d\Omega} = \frac{e^2 \omega^2}{4\pi m c^3} \sin^2\theta$$

polarization is completely in $\hat{\theta}$ direction

6. Again consider a spinless particle of mass m and charge e n the first excited state of a three-dimensional harmonic oscillator characterized by a frequency ω. However, this time assume the charged particle is originally in a state with angular momentum projection $m = +1$ along the z axis. Using the interaction

$$H_{\text{int}} = \vec{j} \cdot \vec{A}/c,$$

and applying the dipole approximation,

(a) Find the decay rate Γ of the first excited state.

(b) Find the differential decay rate $d\Gamma/d\Omega$.

a) Same as (a) from previous

b) $$\langle 0 | \frac{P_x - iP_y}{\sqrt{2}} | n=1, m=1 \rangle = im\omega \langle 0 | \frac{x - iy}{\sqrt{2}} | n=1, m=1 \rangle$$

$$= im\omega \langle 0 | x - iy \{ |n_x = 1\rangle + i |n_y = 1\rangle \}$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}$$

$$\langle 0 | P_x + iP_y | n=1, m=1 \rangle = \langle 0 | P_z | n=1, m=1 \rangle = 0$$

$$\frac{d\Gamma}{d\Omega} = \sum_s \frac{e^2\omega}{2\hbar c\pi^2} \left| \vec{\varepsilon}_s \cdot \left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}} \right) \right|^2$$

$$\hat{\varphi} = -\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y}$$

$$\hat{\theta} = \hat{z}\sin\theta + \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi$$

$$\frac{d\Gamma}{d\Omega} = \frac{e^2\omega}{4\hbar c\pi^2} \left\{ |-\sin\varphi + i\cos\varphi|^2 + \cos^2\theta\, |\cos\varphi + i\sin\varphi|^2 \right\}$$

$$= \frac{e^2\omega}{4\hbar c\pi^2} \left\{ 1 + \cos^2\theta \right\}$$