- An electron is subject to a uniform, time-independent magnetic field $\vec{B} = B\hat{z}$. At $t=0$ the electron is in an eigenstate of $\vec{S}\cdot\hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector lying in the xz-plane that makes an angle β w/ the z-axis.

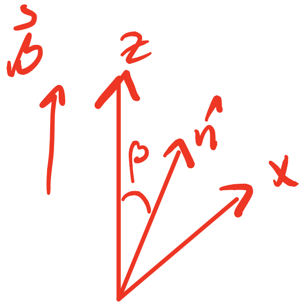
  a) Obtain the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.

  b) Find the expectation value of S_x as a function of time.

  c) Show that the answers make sense in the $\beta=0$ and $\beta=\frac{\pi}{2}$ cases.

- It asks for the evolution in time, this means we need the hamiltonian for the system

$$H = \frac{e}{m_e c} \vec{S} \cdot \vec{B}$$



$$\vec{S} = \frac{\hbar}{2} \vec{\sigma} \quad , \vec{B} = B\hat{z}$$

$$\Rightarrow H = \frac{eB}{m_e c} S_z = \omega S_z$$

$$\Rightarrow E_{\uparrow,\downarrow} = \pm \frac{\omega\hbar}{2}$$

- We then want to write $\vec{S} \cdot \hat{n}$ in terms of eigenstates of the S_z operator so we can use the time evolution opperator

$$\vec{S} \cdot \hat{n} = \cos(\beta/2)|\uparrow\rangle + \sin\frac{\beta}{2}|\downarrow\rangle \equiv |S_n\rangle$$

$$|S_n(t)\rangle = \sum_j e^{-\frac{iE_j t}{\hbar}} |j\rangle\langle j|S_n\rangle$$

$$= e^{-\frac{i\omega t}{2}} \cos(\beta/2)|\uparrow\rangle + e^{\frac{i\omega t}{2}} \sin(\frac{\beta}{2})|\downarrow\rangle$$

- Next we need to find the $|S_x, \uparrow\rangle$ state also in the S_z basis so we can find the overlap between $|S_n\rangle$ and $|S_x\rangle$.

$$|S_x, \uparrow\rangle = \cos(\frac{\pi}{4})|\uparrow\rangle + \sin(\frac{\pi}{4})|\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

- Now we can calculate the probability of $|S_n\rangle$ being measured in the $|S_x,\uparrow\rangle$ state.

$$P(x,\uparrow)(t) = |\langle S_x,\uparrow|S_n\rangle|^2$$

$$= \frac{1}{2}\left|e^{\frac{-i\omega t}{2}}\cos\frac{\beta}{2} + e^{\frac{i\omega t}{2}}\sin\frac{\beta}{2}\right|^2$$

$$= \frac{1}{2}\left(e^{\frac{-i\omega t}{2}}\cos\frac{\beta}{2} + e^{\frac{i\omega t}{2}}\sin\frac{\beta}{2}\right)\left(e^{\frac{i\omega t}{2}}\cos\frac{\beta}{2} + e^{\frac{-i\omega t}{2}}\sin\frac{\beta}{2}\right)$$

$$= \frac{1}{2}\left(\cos^2\frac{\beta}{2} + e^{-i\omega t}\cos\frac{\beta}{2}\sin\frac{\beta}{2} + e^{i\omega t}\cos\frac{\beta}{2}\sin\frac{\beta}{2} + \sin^2\frac{\beta}{2}\right)$$

$$= \frac{1}{2}\left(1 + 2\cos\omega t\cos\frac{\beta}{2}\sin\frac{\beta}{2}\right)$$

$$\boxed{P(x,\uparrow)(t) = \frac{1}{2}\left(1 + \sin\beta\cos\omega t\right)}$$

b) It is a two state system so to get the probability of the $P(x,\downarrow)$ state we just do $1 - P(x,\uparrow)$ because the two probabilities need to sum to one

$$P(x,\downarrow)(t) = 1 - \frac{1}{2}\left(1 + \sin\beta\cos\omega t\right)$$

$$= \frac{1}{2}\left(1 - \sin\beta\cos\omega t\right)$$

- From here to get expectation value we multiply the measurement of S_x by the probability of being in that state and sum them

$$\langle S_x \rangle = P(x,\uparrow)\frac{\hbar}{2} + \left(-\frac{\hbar}{2}\right)P(x,\downarrow)$$

$$= \frac{\hbar}{4}\left(\cancel{1} + \sin\beta\cos\omega t - \cancel{1} + \sin\beta\cos\omega t\right)$$

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2}\sin\beta\cos\omega t}$$

c) - $\beta = 0$ The spin is in the z direction so the expectation value should be zero in x

$$\langle S_x \rangle = \frac{\hbar}{2}\sin(0)\cos\omega t = 0$$

- $\beta = \frac{\pi}{2}$ The spin is in the x direction so at time $t=0$ it should be in the $+\frac{\hbar}{2}$ state.

$$\langle S_x \rangle = \frac{\hbar}{2}\cos\omega(0) = \frac{\hbar}{2}$$