Subject Exam Problem: Fermi Gas

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Problem: Consider a three-dimensional gas of identical spin 1/2 particles of mass m in an L^3 box, with:

$$|\Phi\rangle = \prod_{\alpha,\,|k|_\alpha < k_f} a^\dagger_\alpha |0\rangle$$

summing over all the states with α with momentum $k_\alpha < k_f$. The density-density correlation function in 3D is defined as:

$$C_{s_1,\,s_2}(x_2 - x_1) = 1 - \delta_{s_1,\,s_2}\left[\frac{\frac{1}{|\Delta x|^3}\sin(k_f|\Delta x|) - \frac{k_f}{|\Delta x|^2}\cos(k_f|\Delta x|)}{k_f^3/3}\right]^2$$

(a) Consider the interaction potential $V(\vec{r}) = \alpha\Theta(\beta - |\vec{r}\,|)$, $\alpha < 0$, with the interaction Hamiltonian

$$H_{int} = \frac{1}{2}\sum_{s_1,s_2}\int d^3\vec{r}_1 d^3\vec{r}_2 V(\vec{r}_1 - \vec{r}_2)\Psi^\dagger(\vec{r}_1)\Psi^\dagger(\vec{r}_2)\Psi(\vec{r}_2)\Psi(\vec{r}_1)$$

Calculate the first order energy perturbation per unit volume if all the spins are spin-up.

(b) If one side of the box were shortened to a length of $L/2$, would the magnitude of the energy perturbation increase or decrease? Why?

(c) If the particles spin was allowed to be free now, instead of spin-up, what would the density-density correlation of $C(|\Delta\vec{r}| = 0$ be?

(d) If the particles spin was allowed to be free, would the magnitude of the energy perturbation increase or decrease?

Solution:

Part (a):

With the given correlation function, we can find the average

$$\begin{aligned}\langle H_{int}\rangle &= \frac{1}{2}\sum_{s_1,s_2}\int d^3\vec{r}_1 d^3\vec{r}_2 V(\vec{r}_1-\vec{r}_2)\langle\Phi|\Psi^\dagger(\vec{r}_1)\Psi^\dagger(\vec{r}_2)\Psi(\vec{r}_2)\Psi(\vec{r}_1)|\Phi\rangle \\ &= \frac{1}{2}\sum_{s_1,s_2}\int d^3\vec{r}_1 d^3\vec{r}_2 V(\vec{r}_1-\vec{r}_2)C(\vec{r}_1-\vec{r}_2)\cdot n^2\end{aligned}$$

Here, with only one spin accessible, we have

$$n=\frac{N}{V}=\frac{2(s+1)}{(2\pi)^3}\int\limits_{|k|<k_f} d^3\vec{k}=\overbrace{\frac{1}{(2\pi)^3}}^{\text{one spin}}\frac{4\pi}{3}k_f^3=\frac{k_f^3}{6\pi^2}$$

So

$$\begin{aligned}\langle H_{int}\rangle &= \frac{V}{2}\sum_{s_1,s_2}\int d^3\Delta\vec{r}\; V(\Delta\vec{r})\, C(\Delta\vec{r})\cdot\left(\frac{k_f^3}{6\pi^2}\right)^2 \\ &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\sum_{s_1,s_2}\int d^3\vec{r}\,\alpha\Theta(\beta-|\vec{r}|)\left[1-\delta_{s_1,\,s_2}\left[\frac{\frac{1}{|\vec{r}|^3}\sin(k_f|\vec{r}|)-\frac{k_f}{|\vec{r}|^2}\cos(k_f|\vec{r}|)}{k_f^3/3}\right]^2\right] \\ &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\sum_{s_1,s_2}\int_0^\beta dr 4\pi r^2\,\alpha\left[1-\delta_{s_1,\,s_2}\left[\frac{\frac{1}{|\vec{r}|^3}\sin(k_f|\vec{r}|)-\frac{k_f}{|\vec{r}|^2}\cos(k_f|\vec{r}|)}{k_f^3/3}\right]^2\right]\end{aligned}$$

Now since these particles are all spin-up, we may ignore the sum (as we did when calculating the density) and proceed as

$$\begin{aligned}\langle H_{int}\rangle &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\alpha\int_0^\beta dr4\pi r^2\left(1-\frac{3}{k_f^3}\left(\frac{1}{r^3}\sin(k_f|\vec{r}|)-\frac{k_f}{|\vec{r}|^2}\cos(k_f|\vec{r}|)\right)\right) \\ &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\alpha\left(\frac{4\pi\beta^3}{3}-\frac{3}{k_f^3}4\pi\left(\int_0^\beta dr\frac{1}{r}\sin(k_f r)-k_f\cos(k_f r)\right)\right) \\ &= \frac{V}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\alpha\left(\frac{4\pi\beta^3}{3}-\frac{3}{k_f^3}4\pi\left(\mathrm{Si}(\beta k_f)-\sin(\beta k_f)\right)\right)\end{aligned}$$

By dividing by V, we get the answer of

$$\frac{\langle H_{int}\rangle}{V}=\frac{1}{2}\left(\frac{k_f^3}{6\pi^2}\right)^2\alpha\left(\frac{4\pi\beta^3}{3}-\frac{3}{k_f^3}4\pi\left(\mathrm{Si}(\beta k_f)-\sin(\beta k_f)\right)\right)$$

And we can see the units on this work out, as $[\alpha]=[E]$ and the units on the whole thing are:

$$\left[\left(\frac{k_f^3}{3\pi^2}\right)^2\right][\alpha]\left[\frac{4\pi\beta^3}{3}-\frac{3}{k_f^3}4\pi\left(\mathrm{Si}(\beta k_f)-\sin(\beta k_f)\right)\right]=\frac{1}{[L]^6}[E][L^3]$$

$$\left[\frac{\langle H_{int}\rangle}{V}\right]=\left[\frac{E}{L^3}\right]$$

Part (b):

The magnitude would increase, because the particles would be more correlated in space and thus more of the spherical-well interactions would occur in each unit volume. The fermi energy increases as well.

Part (c):

The density-density correlation would be 1/2, as the correlation with particles of the opposite spin would be 1, and correlation with particles of the same spin is zero. To calculate the total correlation then, we add these together. However, since the density increased by a factor of 1/2, there are twice as many particles and so this number has to be divided by 2, so the total correlation is 1/2.

Part (d):

The magnitude would increase, because the particles are more correlated in space, and the spherical-well interactions can occur between particles with opposite spin. So then, there are twice as many particles to interact with, and so there will be more interactions per unit volume, all of which increase the magnitude of the energy perturbation.