Possible Exam Problem: Chapter 2

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Consider a particle of mass m incident on the following potential:

$$V(x) = -\alpha\delta(x) \tag{1}$$

where

$$\delta(x) = \begin{cases} 0 & x \neq o \\ \infty & x = 0 \end{cases} \tag{2}$$

and where α is a positive constant.

1. How many bounds states are possible?
2. Find an expression for the bound state energy/energies in terms on m and α.
3. Derive the reflection coefficient for particles incoming from the left.
4. Using the result from part C, what is the transmission coefficient?
5. A delta potential can be considered a finite square well with the height taken to the limit of infinity and the width taken to the limit of zero. Find the energy of the bound state of the delta function by treating it as a finite square well. (Inspired by a problem in Griffiths)

Part 1

For this delta potential there is only one bound state.

Proof:

The wave function for x $<$ 0 is given by:

$$\psi_1(x) = Ae^{\kappa x} \tag{3}$$

and the wave function for x $>$ 0 is given by:

$$\psi_2(x) = Be^{-\kappa x} \tag{4}$$

where

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}}. \tag{5}$$

At x = 0 the boundary conditions are:

1. $\psi_1(0) = \psi_2(0)$
2. $\frac{d\psi_1(x)}{dx}\Big|_{x=0} = \frac{d\psi_2(x)}{dx}\Big|_{x=0}$, except at infinite discontinuities.

Given these A = B and

$$\Delta(\frac{d\psi}{dx}) = -B\kappa - B\kappa = -2B\kappa. \tag{6}$$

Also

$$\Delta(\frac{d\psi}{dx}) = -\frac{2m\alpha}{\hbar^2}\psi(0). \tag{7}$$

But since $\psi(0)$ = B, combining Eq. 6 and Eq. 7 yields

$$\kappa = \frac{m\alpha}{\hbar^2}. \tag{8}$$

Since there is only one possible value of kappa, there is only one bound energy.

Part 2

The energy of the only bound state, in terms of m and α is:

$$E = -\frac{m\alpha^2}{2\hbar^2} \tag{9}$$

Proof:

From the previous problem, κ was found to be:

$$\kappa = \frac{m\alpha}{\hbar^2} \tag{10}$$

Then the energy of the bound state is given by:

$$E = -\frac{\hbar^2\kappa^2}{2m} = -\frac{\hbar^2}{2m}(\frac{m\alpha}{\hbar^2})^2 = -\frac{m\alpha^2}{2\hbar^2} \tag{11}$$

Part 3

For scattering, the wave function for x $<$ 0 is:

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx} \tag{12}$$

and the wave function for x $>$ 0 is:

$$\psi_2(x) = Ce^{ikx} + De^{-ikx} \tag{13}$$

Using boundary condition 1, defined in Part 1, A + B must equal C + D. The second boundary condition yields:

$$ik(C - D - A + B) = -\frac{2m\alpha}{\hbar^2}(A + B) \tag{14}$$

or

$$C - D = A(1 + 2i\beta) - B(1 - 2i\beta) \tag{15}$$

where

$$\beta = \frac{m\alpha}{\hbar^2 k}. \tag{16}$$

For a wave incoming from the left, D will be zero.

For the remaining variables, A is the amplitude of the incident wave, B is the amplitude of the reflected wave, and C is the amplitude of the transmitted wave.

Using the boundary conditions yields B and C in terms of A:

$$B = \frac{i\beta}{1 - i\beta}A \tag{17}$$

$$C = \frac{1}{1 - i\beta}A. \tag{18}$$

Finally the reflection coefficient (the fraction of incoming particles that will bounce back from the potential), R, can be defined as:

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1 + \beta^2} \tag{19}$$

Part 4

Given that R = $\frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$ from part C, then the transmission coefficient, T, is:

$$T = 1 - R = \frac{1+\beta^2}{1+\beta^2} - \frac{\beta^2}{1+\beta^2} = \frac{1}{1+\beta^2} \tag{20}$$

Part 5

Finite Square Well

The finite square well is defined as follows:

$$V(x) = \begin{cases} -V_0 & -a < x < a \\ 0 & |x| > a \end{cases} \tag{21}$$

where V_0 is a positive constant.

In the region of x < -a, the potential is zero and Schrodinger's equation gives:

$$\frac{d^2\psi}{dx^2} = \kappa^2\psi \tag{22}$$

where κ is defined in the usual way:

$$\kappa = \frac{\sqrt{-2mE}}{\hbar}. \tag{23}$$

From these equations, the only physically possible wave function is $\psi = Ae^{\kappa x}$.

For the region in the potential well, -a < x < a, Schrodinger's equation reads as:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \longrightarrow \frac{d^2\psi}{dx^2} = -l^2\psi \tag{24}$$

where l is defined to be:

$$l = \frac{\sqrt{2m(E+V_0}}{\hbar} \tag{25}$$

Note: E must be greater than V_0 for a bound state.

The possible wave function in this region is:

$$\psi(x) = Csin(lx) + Dcos(lx) \tag{26}$$

And finally the wave function in the region x > a is:

$$\psi(x) = Be^{-\kappa x} \tag{27}$$

using the same equations as for x < -a.

Since the square well potential does not have an infinite discontinuity, the function and the slope of the wave function must be continuous at all points. Solving for this yields the following equation:

$$\kappa = l * tan(la). \tag{28}$$

From here it is standard to switch notation from κ and l to z and z_0, where z roughly corresponds to the energy and z_0 corresponds to the size of the well.

$$z = la \tag{29}$$

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0} \tag{30}$$

Thus, using this notation:

$$\kappa = l * tan(la) \longrightarrow tan(z) = \sqrt{(\frac{z_0}{z})^2 - 1} \tag{31}$$

The solutions to the above equation are the bound states of the finite square well potential.

Delta Potential from Finite Square Well

From the solution of a finite square well, the following definition is found:

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0} \tag{32}$$

where the width of the square well is 2a.

The area of the square well should be held constant, even as a $\longrightarrow$ 0, so the area of the potential is

$$\omega = 2aV_0 \tag{33}$$

Then $V_0 = \frac{\omega}{2a}$ and

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0} = \frac{a}{\hbar}\sqrt{2m(\frac{\omega}{2a})} = \frac{1}{\hbar}\sqrt{m\omega a} \tag{34}$$

The bound state energies of a finite square well can be found using this equation:

$$tan(z) = \sqrt{(\frac{z_0}{z})^2 - 1} \tag{35}$$

When z approaches zero, tan(z) can be expanded

$$tan(z) = z = \sqrt{(\frac{z_0}{z})^2 - 1} = \frac{1}{z}\sqrt{z_0^2 - z^2} \tag{36}$$

It is also known that from the equations of a finite square well that

$$\kappa^2 a^2 = z_0^2 - z^2 \tag{37}$$

From Eq. 36 and Eq. 37 it can be determined that

$$z^2 = \kappa a \tag{38}$$

and that $z_0 = z$. Therefore $z_0^2 = \kappa a$.
Combining this result with Eq. 34 yields

$$\kappa a = \frac{1}{\hbar^2} m\omega a \longrightarrow \kappa = \frac{m\omega}{\hbar^2} \tag{39}$$

From the finite square well equations, κ is defined as

$$\kappa = \frac{\sqrt{-2mE}}{\hbar} \tag{40}$$

Thus

$$\frac{m\omega}{\hbar^2} = \frac{\sqrt{-2mE}}{\hbar} \longrightarrow E = -\frac{m\omega^2}{2\hbar^2} \tag{41}$$

If ω (the area of the square well) in this problem is equal to α in the definition of the delta potential, then the same result is obtain.